Math 5525 – Homework IV

(Due Monday, April 2)

From the text.

Chapter	Problems
7	1bcde, 3, 5
8	Read this. Covered by the additional problems below

Plus the following additional problems (but problem 6 is optional and will not be graded)

1. Consider the forced, damped oscillator

$$x'' + \delta x' + x = \cos(\omega t)$$

   If $\delta > 0$ we know that all solutions have the same limiting "steady-state" behavior as $t \to \infty$. This problem investigates how that limiting behavior depends on the angular velocity of the forcing, ω, and the damping constant, δ.

   (a) Write the problem as a forced 2×2 linear system $X' = AX + G(t)$. Use the guessing method to find a *periodic*, particular solution $X_p(t)$.

   (b) Let $X_p(t) = (x(t), v(t))$ where $v = x'$. The function $x(t)$ can be put into the form

$$x(t) = A\cos(\omega t - \phi)$$

   for some constants $A > 0$ and ϕ. Call A the *amplitude response* of the oscillator. Find an explicit formula for $A(\delta, \omega)$.

   (c) Plot the function $A(0.1, \omega)$. What angular velocity ω produces the largest amplitude response ? Same for $A(1.0, \omega)$.

2. Use the variation of parameter formula to find the general solution of the forced linear system

$$X' = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ e^t \end{bmatrix}.$$

   Begin by calculating the matrix exponential.

3. For each of the following nonlinear ODEs, find all the equilibrium points and use the linearized ODE to classify them as stable or unstable nodes, stable or unstable spirals, saddles, etc.

   (a)

$$x' = -x + y$$
$$y' = y - 3x^2$$

   (b) $x'' + x' + x^3 - x = 0$. Begin by writing it as a (nonlinear) system.

Continued on the next page

4. Consider the Lorenz ODE in $\mathbf{R}^3$

$$\begin{aligned}\dot{x} &= y - x\\ \dot{y} &= ax - y - xz\\ \dot{z} &= xy - z\end{aligned}$$

where $a > 0$ is a parameter. Find the equilibrium points, find the linearized ODE at these points and the corresponding eigenvalues. At which value of a is there a bifurcation of the number of equilibrium points ? At which values of a is there a change in the stability of an equilibrium point?

5. Consider the following autonomous ODE in $\mathbf{R}^2$

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= y + x^2\end{aligned}$$

a. Use elementary methods to find a formula for the flow $\phi(t,(x_0,y_0))$ and verify that the flow axioms hold:

$$\phi(0,(x_0,y_0)) = (x_0,y_0) \qquad \phi(s+t,(x_0,y_0)) = \phi(s,\phi(t,(x_0,y_0))).$$

b. Find the and sketch the set

$$W^s(0,0) = \{(x_0,y_0) : \phi(t,(x_0,y_0)) \to (0,0) \text{ as } t \to \infty\}$$

of all initial conditions whose orbits converge to the origin as $t \to \infty$. This is called the stable curve of the equilibrium point $(0,0)$. Similarly, find and sketch the unstable curve,

$$W^u(0,0) = \{(x_0,y_0) : \phi(t,(x_0,y_0)) \to (0,0) \text{ as } t \to -\infty\}.$$

6. (Optional problem. No need to hand this in.) Verify the theorem of section 7.4 for the differential equation below. First find the formula for the flow of the system, $\phi(t,X)$ and compute the Jacobian matrix $D\phi(t,X_0)$ for initial position $X_0 = (1,0)$. Next write down the variational equations along the solution $\phi(t,X_0)$ and solve them. Finally, compare the two results to verify the theorem.

$$\begin{aligned}x' &= -x\\ y' &= y - 3x^2\end{aligned}$$