Math 5535 – Homework I
Due Monday, September 25

1. For each of the following maps \( f(x) \), find all of the fixed points and analyze their stability.
   i. \( f(x) = \frac{5}{2}x(1 - x) \)
   ii. \( f(x) = \frac{3}{2}x(1 - x) \)
   iii. \( f(x) = \frac{x}{2} + \frac{1}{x} \)
   iv. \( f(x) = \frac{1 + x}{2} \)
   v. \( f(x) = \sqrt{2} + x \)
   vi. \( f(x) = \sin x \). Hint: start with a cobweb plot; then write down a proof.

2. Evaluate the limit of the following sequence:
   \[
   \sqrt{2} \quad \sqrt{2 + \sqrt{2}} \quad \sqrt{2 + \sqrt{2 + \sqrt{2}}} \quad \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \ldots
   \]
   Explain your answer carefully. Hint: Consider \( f(x) = \sqrt{2 + x} \).

3. For \( f(x) = x^2 - \frac{3}{2} \), find all of the points of minimal period 2 and analyze their stability. (See hint for problem 9 below.)

4. Let \( f(x) = -\frac{3}{2}x^3 - \frac{3}{2}x^2 + x + 1 \). Show that \( x_0 = 0 \) is a periodic point. Find its minimal period and determine its stability.

5. The Covering Interval Fixed Point Theorem. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous map and let \( I = [a, b] \) be a closed interval such that \( f(I) \supset I \). Prove that \( f \) has at least one fixed point in \( I \). Hint: Since \( f(I) \supset I \) there exist points \( c, d \in I \) with \( f(c) = a \) and \( f(d) = b \). Consider the graph of \( f \) over the interval between \( c \) and \( d \). There are several cases corresponding to whether \( c < d \) or \( d < c \) and whether or not \( c \) or \( d \) equal \( a \) or \( b \).

6. Let \( S \) be a metric space with distance function \( d \) and let \( f : S \to S \) be a continuous map. If \( \bar{x} \in S \) is a fixed point for \( f \) then it is also fixed for the iterates of \( f \), for example, for the second iterate \( f^2(x) \). Show that if \( \bar{x} \) is asymptotically stable for the map \( f^2 \) then it is also asymptotically stable for \( f \). Hint: According to the definition, you need to prove both stability and attractivity for \( f \) whereas you are given those two things for the second iterate \( f^2 \). Consider the even iterates \( f^{2n}(x) \) and odd iterates \( f^{2n+1}(x) \) separately and use the fact that \( f^{2n+1}(x) = f^{2n}(f(x)) \).
7. Let \( f(x) = 1 + \tanh(x) = \frac{2}{1+e^{-2x}} \). Prove that \( x \) has a globally attracting fixed point \( \bar{x} \in [1,2] \), i.e., \( \bar{x} \) is a fixed point such that for every initial state \( x_0 \in \mathbb{R} \), we have \( x_n \to \bar{x} \) as \( n \to \infty \). Hint: start by showing that \( f \) gives a contraction map of \([0, \infty)\) into itself.

8. Let \( f(x) = \frac{x}{2} + \frac{1}{x} \) be the Newton function for finding roots of \( g(x) = x^2 - 2 \). Show that for every initial condition \( x_0 \in (0, \infty) \) we have \( x_n \to \sqrt{2} \). What happens if \( x_0 \in (-\infty, 0) \)?

9. Consider the quadratic maps \( f(x) = x^2 - c \) where \( c \) is a constant. Determine for which values of \( c \) there exists an orbit of minimal period two. Find the two points \( x_0, x_1 \) of the orbit and the compute the multiplier \( \mu = f'(x_0)f'(x_1) \). For which values of \( c \) is the orbit an attractor (i.e., asymptotically stable)? Hint: the formula below should be useful:

\[
x^4 - 2cx^2 + c^2 - c = (x^2 - x - c)(x^2 + x + 1 - c).
\]

10. For the decimal shift map \( f(x) = 10x \mod 1 \) find all the periodic points of minimal period 2 such that the periodic orbits are entirely contained in the interval \([0, 0.3)\). For each point give both its repeating decimal expansion and its representation as a fraction.