10, page 25. Since \( f(I) \supset I \) there are points \( c, d \) in \( I \) which take on the values \( a, b \). If \( c = a \) or \( d = b \) then we have a fixed point at one of the endpoints so we are done. Otherwise the function \( g(x) = f(x) - x \) is negative at \( x = c \) and positive at \( x = d \). Apply the intermediate value theorem on the interval \([c, d]\) or \([d, c]\) according to whether \( c < d \) or \( d < c \) to conclude that there must be a fixed point of \( f(x) \) between \( c \) and \( d \).

16, page 35. Stability: Let \( \epsilon > 0 \) be given. By stability under \( f^2 \) we can find a \( \delta' > 0 \) such that if \( d(x_0, \bar{x}) < \delta' \) then \( d(f^{2n}(x_0), \bar{x}) < \epsilon \) for all \( n \). We need to arrange that the odd iterates also remain within \( \epsilon \). But since \( f^{2n+1}(x_0) = f^{2n}(f(x_0)) = f^{2n}(x_1) \) it is enough to get \( d(x_1, \bar{x}) < \delta' \) (then we can apply the inequality for \( f^{2n} \) above with \( x_0 \) replaced by \( x_1 \)). Since \( f(x) \) is continuous, there is \( \delta > 0 \) such that if \( d(x_0, \bar{x}) < \delta \) then \( d(f(x_0), f(\bar{x})) < \delta' \), i.e., \( d(x_1, \bar{x}) < \delta' \) which is what we want. Making \( \delta \) smaller if necessary, we can assume \( \delta \leq \delta' \). Then if \( d(x_0, \bar{x}) < \delta \) we will have both \( d(x_0, \bar{x}) < \delta' \) and \( d(x_1, \bar{x}) < \delta' \) and so both \( d(f^{2n}(x_0), \bar{x}) < \epsilon \) and \( d(f^{2n+1}(x_0), \bar{x}) < \epsilon \).

Attractivity: We are given that there is an \( \alpha > 0 \) such that if \( d(x_0, \bar{x}) < \alpha \) then \( x_{2n} = f^{2n}(x_0) \to \bar{x} \). To show that the odd iterates \( x_{2n+1} \) also converge to \( \bar{x} \), just use continuity of \( f \) to see that

\[
\lim f^{2n+1}(x_0) = \lim f(f^{2n}(x_0)) = f(\lim x_{2n}) = f(\bar{x}) = \bar{x}.
\]

Since the even and odd subsequences converge to the same limit \( \bar{x} \) it follows that the whole sequence converges to \( \bar{x} \).

Extra problem 1. One can check that \( f(1) - 1 > 0 \) and \( f(2) - 2 < 0 \) so there is a fixed point \( \bar{x} \in [1, 2] \). Here are two approaches to the problem of global attraction.

The first is similar to the argument for the cosine function. We want to use the fact that the map is (almost) a contraction map. The derivative satisfies \( |f'(x)| = \text{sech}^2(x) = \frac{4}{(e^x + e^{-x})^2} \leq 1 \) but it is equal to 1 when \( x = 0 \) so we do not immediately get a distance-contraction factor \( \lambda < 1 \). But we can get such a factor on any interval which avoids \( x = 0 \). For example, consider \( I = [1, 2] \). Looking at the graph of \( f \) it is clear that for any \( x \geq 0 \), we have \( 1 \leq f(x) < 2 \). In particular, \( f \) maps \( I \) into itself, \( f: I \to I \). And on \( I \) we have \( |f'(x)| \leq \text{sech}^2(1) < 1 \) so \( f: I \to I \) is a contraction map. It follows from a theorem in class that the fixed point in \( I \) is asymptotically stable and attracts the whole interval \( I \). To handle general initial conditions \( x_0 \in \mathbb{R} \) note that for
any $x_0$, the first iterate $x_1 = f(x_0) \in [0, 2]$ and then the second iterate $x_2 \in [1, 2]$. It follows that these orbits are also attracted to $\bar{x}$.

Another approach is inspired by the shape of the graph and some of the cobweb plot arguments we discussed. Note that the derivative satisfies $0 < f'(x) \leq 1$ and is strictly less than 1 away from the origin. So $f(x)$ is strictly increasing. It covers the range $0 < f(x) < 2$ as $x$ runs from $-\infty$ to $\infty$. The value at $x = 0$ is $f(0) = 1$. Since the slope is strictly less than 1 except at $x = 0$, the function $f(x) - x$ is strictly decreasing and it follows that there can be only one fixed point, the point $\bar{x} \in [1, 2]$ mentioned above. It follows from this analysis that on the interval $(-\infty, \bar{x})$ we have $x < f(x) < x$ and on the interval $(\bar{x}, \infty)$ we have $\bar{x} < f(x) < x$. Therefore if $x_0 \in (-\infty, \bar{x})$, the orbit $x_n = f^n(x_0)$ remains in this interval for all $n$ and the sequence is monotonically increasing. Since it is bounded above by $\bar{x}$ is has a limit which must be a fixed point and since $\bar{x}$ is the only fixed point, we have $x_n \to \bar{x}$. Similarly, if $x_0 \in (\bar{x}, \infty)$, the orbit $x_n = f^n(x_0)$ remains in this interval for all $n$, the sequence is monotonically decreasing and must converge to $\bar{x}$.

Extra problem 2. Setting $f^2(x) - x = 0$ and factoring as in the hint gives one factor representing fixed points and another representing points of minimal period 2. Solving the latter gives two points $x_0, x_1 = -1 \pm \sqrt{4c-3}$. These are real and distinct for $c > \frac{3}{4}$. The multiplier is $\mu = f'(x_0)f'(x_1) = 4(1 - c)$. We certainly have asymptotic stability if $|\mu| < 1$ and instability if $|\mu| > 1$. The first inequality can be written $1 < \mu < 1$ which simplifies to $\frac{3}{4} < c < \frac{5}{4}$ so these values of $c$ give asymptotic stability. We have instability for $c > \frac{5}{4}$. This is what I intended you to do for this problem but strictly speaking one should check the borderline case $c = \frac{5}{4}$ which is subtler (call it extra credit!)

Looking at the graph of $f^4(x)$ near the points $x_0, x_1$ shows that they are attracting fixed points (similar to problem 1vi). From problem 6 we can conclude they are also attracting fixed points for $f^2$. By definition, this means the periodic two orbit $\{x_0, x_1\}$ is an attractor for $c = \frac{5}{4}$. 