9, page 80. I will change notation so the logistic map is \( f_a(x) = ax(1-x) \). Here are two ways to solve the problem. First a direct proof that all of the periodic orbits are repellers. For \( a > 2 + \sqrt{5} \approx 4.23607 > 4 \) the graph of \( f_a \) on \([0,1]\) extends beyond the interval \([0,1]\). In fact, the highest point on the graph has height \( f_a(1/2) = a/4 > 1 \). Using cobweb plots we saw that any initial point \( x_0 \) outside the interval \([0,1]\) has \( x_n \to -\infty \), so any periodic points must lie in \([0,1]\). From the graph we see that there are two subintervals of \([0,1]\) in which \( f(x) \in [0,1] \) and so any periodic points must always remain in these two intervals. Some algebra involving solving quadratic equations that for \( a > 2 + \sqrt{5} \) the slope of the graph on these two subintervals satisfies \(|f'(x)| > 1 \). So the multiplier of any periodic orbits moving between these two intervals satisfies \( \mu = |f'(x_0)f'(x_1)\ldots f'(x_{q-1})| > 1 \) and all of these orbits must be repellers. There is also an indirect proof using Singer’s theorem which actually works for all \( a > 4 \). After making the observations about the graph, note that orbit of the critical point \( x_0 = 1/2 \) leaves the interval \([0,1]\) after one time step and therefore tend to \( -\infty \). So \( 1/2 \) cannot be in the basin of attraction of any periodic attractor. Such a basin, if it existed, would be bounded since it would be in \([0,1]\). Singer’s theorem shows that such an attractor cannot exist.

2, page 92. The map \( R_\mu(x) = \mu x e^{-x} \) has fixed points at \( x = 0 \) and \( x = \ln(\mu) \). The multiplier at \( x = \ln(\mu) \) is \( R'(x) = 1 - \ln(\mu) \) and for \( \mu = e^2 \) the multiplier is \( -1 \) signaling a possible period-doubling bifurcation. We just need to check some conditions on higher derivatives to be sure. The theorem in the book does not apply since the fixed points is moving as the parameter changes. A theorem from class shows that we need to check two conditions. Using subscript notation for partial derivatives, the two conditions are: \( A = R_{\mu x} + \frac{1}{2}R_{xx} \neq 0 \) and \( B = -R_{xxx} - \frac{3}{2}(R_{xx})^2 \neq 0 \) and it is straightforward to check these at the bifurcation point.

Extra problem 2. The equation \( f^3(1/2) = (a - 1)/a \) is a rational function involving only the parameter \( a \). Using Mathematica to Factor it gives the equation \( (a-2)^4(a+2)(a^3-2a^2-4a-8) = 0 \). The last factor is negative when \( a = 3.5 \) and positive when \( a = 4 \) so there is a root \( a \in [3.5, 4] \). For this value of \( a \), the critical orbit cannot be attracted to a periodic attractor since it lands on a repelling fixed point. Singer’s theorem shows that no attractors can exist.
Extra problem 3. For any odd function, \( f(-x) = -f(x) \). Setting \( x = 0 \) gives \( f(0) = -f(0) \) which implies \( f(0) = 0 \). So 0 is a fixed point. If \( x_0 \neq 0 \) and \( f(x_0) = -x_0 \) then \( f^2(x_0) = f(f(x_0)) = f(-x_0) = -f(x_0) = -(x_0) = x_0 \) so \( x_0 \) is a period 2 point. It’s not fixed since \( f(x_0) = -x_0 \neq x_0 \) so it has minimal period 2. Now consider \( q \geq 2 \). The \( q \)-th iterate \( f^q(x) \) is also an odd function (as is easy to show). So the same argument shows that if \( f^q(x_0) = -x_0 \) then \( f^{2q}(x_0) = x_0 \) and that \( x_0 \) is not fixed by \( f^q \). To see that \( 2q \) is the minimal period suppose that there is some \( k < 2q \) such that \( f^k(x_0) = x_0 \). \( k \) must divide \( 2q \) but it can’t divide \( q \) since \( q \) is not a period. The only other possibility is that \( q \) is odd and \( k = 2 \) so \( f^2(x_0) = x_0 \) and \( f^q(x_0) = -x_0 \). Writing \( q = 2 * m + 1 \) we get \( -x_0 = f^q(x_0) = f(f^{2m}(x_0)) = f(x_0) \). But \( q \geq 2 \) was supposed to be the smallest exponent with \( f^q(x_0) = -x_0 \) so this is a contradiction. Note: this makes problem 4 easier since you can find period 2 points by solving \( f(x) = -x \) instead of \( f^2(x) = x \).

Extra problem 6. Suppose the decimal expansion of \( \bar{x} \) is \( 0.d_1d_2\ldots \). Then \( Ev(\bar{x}) \) consists of all decimals with this “tail”, that is, \( x = 0.e_1e_2\ldots e_n d_1d_2\ldots \) where \( e_1,\ldots, e_n \) is any finite sequence of digits. Given any \( y \in [0,1) \) and any \( \epsilon > 0 \), choose \( n \) such that \( 10^{-n} < \epsilon \). If \( y = 0.e_1e_2\ldots \) let \( x = 0.e_1e_2\ldots e_n d_1d_2\ldots \). Then \( x \in Ev(\bar{x}) \) and \( d(x,y) \leq 10^{-n} < \epsilon \).