2. The fractal contains three copies of itself at scale \( \lambda = \frac{1}{2} \). One is just scaled, one is scaled and translated and one is scaled, rotated and translated. The similarities are

\[
F_0(z) = \frac{1}{2} z, \quad F_1(z) = \frac{1}{2} z + \frac{1}{2}, \quad F_0(z) = \frac{1}{2} e^{i \pi / 4} z + \frac{1 + i}{2}.
\]

The dimension is

\[
\dim_s(A) = -\frac{\log 3}{\log \frac{1}{2}} = \frac{\log 3}{\log 2}.
\]

3. The similarities of the line \( F_0(x) = \frac{1}{4} x \) and \( F_1(x) = \frac{1}{2} x + \frac{1}{2} \) will map the unit interval \([0, 1]\) onto the intervals used in the first step of the construction. Then applying the same two maps to these intervals gives the four intervals at step two of the construction, and so on. Thus we want the attractor of the iterated function system \( F = \{F_0, F_1\} \).

The dimension will be the unique positive solution of

\[
\frac{1^d}{4} + \frac{1^d}{2} = 1.
\]

If we set \( x = \frac{1}{2} \) this becomes a quadratic equation \( x^2 + x - 1 = 0 \). Take the positive root of this and then solve for \( d \) and simplify to get

\[
d = \frac{\log 2 - \log(\sqrt{5} - 1)}{\log 2}.
\]

4. \( K \times K \) can be constructed starting from the unit square by removing horizontal and vertical middle third strips to obtain 4 small squares and then repeating this construction on the small squares to get 16 squares, etc. Alternatively the unit square can be mapped onto the four small squares using similarities such as

\[
\frac{1}{3} z, \quad \frac{1}{3} z + \frac{1}{3}, \quad \frac{2}{3} z, \quad \frac{1}{3} z + \frac{2 + 2i}{3}, \quad \frac{2i}{3} z + \frac{2 + 2i}{3}.
\]

This gives similarity dimension \( \dim_s(K \times K) = \log 4 / \log 3 \).

To show that \( \dim_t(K \times K) = 0 \) let \( z \in K \times K \) and \( \epsilon \) be given. We need to show that there exists a neighborhood \( \tilde{U} \) of \( z \) in \( \mathbb{R}^2 \) inside \( B_\epsilon(x) \) whose boundary satisfies

\[
\partial \tilde{U} \cap K \times K = \emptyset.
\]

Choose \( n \) such that \( \frac{\sqrt{2}}{3^n} < \epsilon \). Let \( x \in K \times K \). Then \( x \) is in one of the \( 4^n \) small squares of edge size \( 1/3^n \) used in the construction. The choice of \( n \) guarantees that this closed square fits inside \( B_\epsilon(x) \) even if
$x$ is at a corner of the square. For $\tilde{U}$ we can choose a square open set slightly bigger than the closed square containing $x$ but still fitting inside the $\epsilon$ ball. If this open square is small enough it will not intersect any of the other squares used in the construction of $K \times K$. Also, the boundary of the open square is outside the square containing $x$, so we have $\partial \tilde{U} \cap K \times K = \emptyset$.

5.i. By definition $D(A, B) = \max(\max_{x \in A} d(x, B), \max_{x \in B} d(x, A))$. Now the point of $A$ farthest from $B$ is $x = 0$ and we have $d(0, B) = 3$. The point of $B$ farthest from $A$ is $x = 4$ and we have $d(x, A) = 2$. Thus $D(A, B) = \max(3, 2) = 3$.

iii. The point of $[0,1]$ farthest from the Cantor set is $x = \frac{1}{2}$ which gives $\max_{x \in A} d(x, B) = \frac{1}{6}$. Since $B \subset A$ we have $\max_{x \in B} d(x, A) = 0$ so $D(A, B) = \frac{1}{6}$.

6.i. The intervals used in the construction are $[0, \frac{1-\sigma}{2}]$, $[\frac{1+\sigma}{2}, 1]$ each of length $\lambda = \frac{1-\sigma}{2}$. So $K_\sigma$ is the attractor of an iterated function system of $N = 2$ maps with contraction factor $\lambda$. Thus $\dim_s(K_\sigma) = \frac{\log 2}{\log \frac{2-\log(1-\sigma)}{2}}$.

We have $\dim_s(K_\sigma) \to 0$ as $\sigma \to 1$ and $\dim_s(K_\sigma) \to 1$ as $\sigma \to 0$. It’s continuous so it takes all values in between.

ii. Set $\dim_s(K_\sigma) = \frac{1}{2}$ and solve for $\sigma$ to find $\sigma = \frac{1}{2}$. 