Solutions to selected problems

7-(a):
If \( n \) is an even integer, then \( n = 2k \) for some integer \( k \). \( n^2 = 4k^2 \), and \( 4 \mid 4k^2 \) since \( k^2 \) is integer . So, \( 4 \mid n \).

7-(b):
If \( n \) is an even integer, then \( n = 2k \) for some integer \( k \). \( n^3 = 8k^3 \), and \( 8 \mid 4k^3 \) since \( k^3 \) is integer . So, \( 8 \mid n \).

7-(c):
If \( n \) is an odd integer, then \( n = 2k + 1 \) for some integer \( k \).
\( n^3 = 8k^3 + 12k^2 + 6k + 1 \), and
\( 2n^3 = 16k^3 + 24k^2 + 12k + 2 = 8(2k^3 + 3k^2 + k) + (4k + 2) \). Suppose \( 8 \mid 2n^3 \), then \( 8 \mid 4k + 2 \), then \( 4 \mid 2k + 1 \), which is impossible since \( 2k + 1 \) is odd.

7-(d):
Suppose \( 2^\frac{1}{2} \) is rational, then we can write \( 2^\frac{1}{2} = \frac{p}{q} \) where \( (p, q) = 1 \), \( p, q \) are integers.
Then we have \( 2 = \frac{p^2}{q^2} \), or \( 2q^2 = p^2 \). Therefore, \( p^2 \) is even and hence \( p \) is even too.
We can write \( p = 2m \), therefore we get \( 2q^2 = 4m^2 \). Therefore, \( 8 \) divide \( 2q^2 \).
But by (c), we know \( q \) must be even. This contradicts to the fact that \( (p, q) = 1 \).

12-(a):
Given \( \varepsilon > 0 \), if \( \forall s \in S \), \( s \leq b - \varepsilon \), then \( b - \varepsilon \) is a upper bound for \( S \) and \( b - \varepsilon < b \), which contradict to the fact that \( b \) is the least upper bound. So it must exist some \( s \in S \) such that \( b - \varepsilon < s \). And such \( s \leq b \) is trivial since \( b \) is an upper bound of \( S \).

12-(b):
No. Let \( S = \{ 1 \} \), then \( b = 1 \). Taking \( \varepsilon = 1 \), we can not find any element \( s \in S \) such that \( 0 < s < 1 \).

12-(c):
If \( x = A \mid B \) is a cut in \( \mathbb{Q} \), by the result proved in class, we have
\( A = \{ r \in \mathbb{Q} \mid r < x \} \), and \( B = \{ r \in \mathbb{Q} \mid r \geq x \} \). Therefore, \( x \) is an upper bound of \( A \). Next we show that it is the least one. Suppose not, say \( y < x \), \( y \) is an upper bound for \( A \). By completeness, we can always find \( z \in \mathbb{Q} \) such that \( y < z < x \), but then \( z \in A \), which contradict to the assumption that \( y \) is a upper bound of \( A \).

25:
\( A = \{ x \in \mathbb{R} : a \leq x \leq b \} \), \( B = \{ y \in \mathbb{R} : \exists s, t \in [0, 1], s + t = 1, \text{and } y = sa + tb \} \).
(1) : \( A \subseteq B \)
If \( x \in A \), then \( a \leq x \leq b \). Write \( x = a + t(b - a) \), where \( t = \frac{x - a}{b - a} \). Then
\( x = (1 - t)a + tb \), where \( (1 - t) + t = 1 \) and \( t, 1 - t \in [0, 1] \), i.e. \( x \in B \).
(2) : \( B \subseteq A \)
If \( y \in B \), then \( \exists s, t \in [0, 1], s + t = 1, \text{and } y = sa + tb \). So
\( y = (1 - t)a + tb = a + t(b - a) \). Since \( 0 \leq t \) and \( a \leq b \), we get \( a \leq y \). Also
\( y = sa + (1 - s)b = b + s(a - b) \). Since \( 0 \leq s \) and again \( a \leq b \), we have \( y \leq b \).
Therefore, \( y \in A \).
32:
Since each \( B_k \) is countable, it is either denumerable or finite. If every \( B_k \) is
denumerable. Then by Corollary 18, the union is denumerable and hence
countable. If some \( B_k \)'s are finite, then we extend them to \( C_k \) that are
denumerable. Also, let \( B_k = C_k \) if they are already denumerable. Then use
Corollary 18 again, the unions of \( C_k \) is denumerable. Therefore the union of \( B_k \)
are at most denumerable and hence countable.
If no \( B_k \) are denumerable, and elements in them repeat in some way or only
finitely many of them are nonempty, the union can be finite.

39-(a):
If \( f \) is uniformly continuous on \( D \), given \( \varepsilon > 0 \), then \( \exists \delta > 0 \) such that \( \forall x, y \in D \)
and \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \varepsilon \). So, for any \( x \in D \) and \( \varepsilon > 0 \), just pick
such \( \delta > 0 \), then they satisfy the requirement for \( f \) being continuous at \( x \).
Therefore, \( f \) is continuous on \( D \).
Conversely, it is not true that continuity imply uniform continuity. The function
\( \frac{1}{x} \) on \((0, 1)\) is a counterexample. First, it is continuous on \((0, 1)\) since both
\( \sin(x) \) and \( \frac{1}{x} \) are continuous on \((0, 1)\) and the composition of continuous functions
is continuous. Given \( \varepsilon = 1 \), for any \( \delta > 0 \), there must exist \( n \) and \( m \) large enough
such that the distance between \( \frac{2}{\pi + 4n\pi} \) and \( \frac{2}{3\pi + 4m\pi} \) is less than \( \delta \), but their
difference of their value of \( f \) is \( 2 > 1 \), which says that \( \frac{1}{x} \) is not uniformly
continuous on \((0, 1)\).

39-(b):
Given \( \varepsilon > 0 \), take \( \delta = \varepsilon/2 \), then if \( |x - y| < \delta \), then \( |x - y| < \varepsilon/2 \), i.e.
\( 2x - 2y < \varepsilon \). Therefore, \( f(x) = 2x \) is uniform continuous on \((-\infty, +\infty)\).

39-(c):
Claim that \( f(x) = x^2 \) is not uniform continuous on \((-\infty, +\infty)\). Given \( \varepsilon = 1 \), for
any \( \delta > 0 \), take any \( x \) such that \( x > \frac{1}{2} \left( \frac{2}{3} - \frac{\delta}{2} \right) \), then we have the distance of \( x \) and
\( x + \frac{\delta}{2} \) is less than \( \delta \), however the distance between \( x^2 \) and \( (x + \frac{\delta}{2})^2 \) is \( > 1 \).

16 (not graded):
Given a real number \( x \) the problem outlines an algorithm producing a sequence of
integers \( N \), \( x_1 \), \( x_2 \), \ldots. If we use \( [x] \) to denote the greatest integer function then the
algorithm can be described as follows. Given \( x \) define \( N = [x] \) and then let \( u_1 = 10(x - N) \). At the next step we take \( x_1 = [u_1] \) and let \( u_2 = 100(x - N - \frac{x_1}{10}) = 10(u_1 - x_1) \). At each successive stage we have \( x_k = [u_k] \) and \( u_k = 10^k(x - N.x_1.x_2\ldots x_{k-1}) \)
where here and below we will use the standard decimal notation for finite sums
\[
N.x_1x_2\ldots x_k = N + \frac{x_1}{10} + \ldots + \frac{x_k}{10^k}.
\]

a. By definition of the greatest integer function, we have \( 0 \leq x - N < 1 \), so
\( 0 \leq u_1 < 10 \). Hence \( x_1 = [u_1] \in \{0, 1, \ldots, 9\} \). Similarly, at each stage we have
\( u_{k+1} = 10(u_k - x_k) \in [0, 10) \) so \( x_{k+1} \in \{0, 1, \ldots, 9\} \).
b. Using the definition of the greatest integer function we have \( x_k \leq u_k < x_k + 1 \) for all \( k \). From the formula for \( u_k \) this gives
\[
N.x_1 x_2 \ldots x_k \leq x < N.x_1 x_2 \ldots x_k + \frac{1}{10^k}.
\] (1)

In fact, the algorithm starting from \( x \) produces the sequence of digits \( N,x_1,\ldots \) if and only if (1) holds for all \( k \).

Now suppose that all of the digits \( x_l, l \geq k \) are \( x_l = 9 \). From (1) we have \( x < N.x_1 x_2 \ldots x_k + \frac{1}{10^k} \). From (1) with \( k \) replaced by \( k + n, n > 0 \) we have \( x \geq N.x_1 x_2 \ldots x_k.9999999 \) where there are \( n \) 9’s. It follows that \( u_{k+1} = 10^{k+1}(x - N.x_1 x_2 \ldots x_k) \) satisfies
\[
9.999999 \leq u_{k+1} < 10
\]
no matter how many 9’s appear on the left side. But this is impossible since any number any number \( u_{k+1} < 10 \) we can choose \( n \) such that \( \frac{1}{10^n} < 10 - u_{k+1} \) and so \( u_k < 10 - \frac{1}{10^n} = 9.99\ldots 9 \). So we have a contradiction.

c. Now given any sequence of digits \( N,x_1,\ldots \) which does not end in an infinite string of 9’s we need to find a real number \( x \) for which the inequalities (1) hold for all \( k \). Let
\[
x = lub\{N,N.x_1, N.x_1 x_2,\ldots, N.x_1 \ldots x_k,\ldots\}.
\] (2)

Since \( x \) is an upper bound for these numbers it will satisfy the lower inequality in (1) for all \( k \). Also, for each \( k \) the number \( N.x_1 \ldots x_k + \frac{1}{10^k} \) is an upper bound for the set appearing in (2). This follows from the fact that the digits are all at most 9 so that for any \( n > 0 \) we have by elementary arithmetic that
\[
N.x_1 \ldots x_{k+n} \leq N.x_1 \ldots x_k 9999999 < N.x_1 \ldots x_k + \frac{1}{10^k}
\]
where there are \( n \) 9’s. In fact, we can choose \( n > 0 \) so \( x_{k+n} \neq 9 \) and let \( b_n = N.x_1 \ldots x_k 9999999 \) where there are \( n \) 9’s. Then again by elementary arithmetic, we have
\[
N.x_1 \ldots x_{k+n'} < N.x_1 \ldots x_{k+n} + \frac{1}{10^{k+n}} \leq b_n
\]
for all \( n' \geq n \). Since the sequence in (2) is increasing, \( b_n \) is an upper bound for the whole sequence and hence also for \( x \). Therefore
\[
x \leq b_n < N.x_1 \ldots x_k + \frac{1}{10^k}
\]
holds and \( x \) does indeed have the given decimal expansion.

33-(a) (not graded):
It suffices to consider the intervals \([0,1], [0,1) \) and \((0,1) \) since there are linear bijections between these intervals and the corresponding ones with endpoints \( a < b \).

It is clear that \([0,1]\) can be decomposed into a disjoint union of denumerably many half-open intervals \( I_0, I_1,\ldots \) where \( I_0 = [0,1/2), I_1 = [1/2,3/4),\ldots, I_n = [1-1/2^n,1-1/2^{n+1}) \). We can also decompose \((0,1)\) into disjoint half-open intervals: \( J_0 = [1/2,1), J_1 = [1/4,1/2),\ldots, J_n = [1/2^{n+1},1/2^n) \). Define a bijection between \( f : [0,1) \rightarrow (0,1) \) by sending each \( I_n \) bijectively onto \( J_n \) (as both are half-open intervals, this is no problem). \( f \) can be extended to a bijection \( g : [0,1] \rightarrow (0,1) \) be mapping the extra point \( x = 1 \) to itself. So we have
\[
[0,1] \sim (0,1] \sim [0,1) \sim (0,1)
\]
where the middle bijection can be done with a linear map.