Answers for the homework assignment 2 of Math 5615H

3:
Let \( S = \{ p \} \) be a set of single point \( p \in M \), where \( M \) is a metric space. Assume \( \{ a_n \} \) is a sequence in \( S \), then \( a_n = p \) for all \( n \), and \( a_n \to p \in S \), which means that \( S \) contains all its limits. Therefore \( S \) is a closed subset of \( M \). Finite set of points is also closed since the finite union of closed set is closed.

6-(a):
Since \( T \) contains \( S \), \( \bar{T} \) contains \( T \), therefore \( \bar{T} \) contains \( S \). Also since \( \bar{T} \) is closed, and by definition \( \bar{S} \) is the intersection of all closed sets containing \( S \), we have \( S \subset \bar{T} \).

6-(b):
Since \( S \) is contained in \( T \), \( \text{int}(S) \) is contained in \( S \), therefore \( \text{int}(S) \) is contained in \( T \). Also since \( \text{int}(S) \) is open, and by definition \( \text{int}(T) \) is the union of all open sets which is contained in \( T \), we have \( \text{int}(S) \subset \text{int}(T) \).

14-(a):
If \( p \) is a limit of \( S \), then there exist \( a_n \in S \) such that \( a_n \to p \). Suppose \( \text{dist}(p,S) = \inf \{ d(p,s) : s \in S \} = \alpha > 0 \), then \( d(p,s) \geq \alpha \) for all \( s \in S \). However, for \( n \) large enough, we have \( d(p,a_n) < \alpha \), which lead to a contradiction. Therefore we must have \( \text{dist}(p,S) = 0 \).
On the other hand, if \( \text{dist}(p,S) = \inf \{ d(p,s) : s \in S \} = 0 \), given \( n \in \mathbb{N} \) there exist \( a_n \in S \) such that \( d(p,a_n) < \frac{1}{n} \). Therefore we have \( a_n \to p \) and \( p \) is a limit of \( S \).

14-(b):
Let \( p, q \in M \) and \( s \in S \), we have \( d(p,s) \leq d(p,q) + d(q,s) \). Therefore for any \( s \in S \) we get \( \text{dist}(p,S) \leq d(p,q) + d(q,s) \). Therefore, \( \text{dist}(p,S) - d(p,q) \) is a lower bound for \( \{ d(q,s) : s \in S \} \). Therefore we have \( \text{dist}(p,S) - d(p,q) \leq \inf \{ d(q,s) : s \in S \} = \text{dist}(q,S) \). That is \( \text{dist}(p,S) - \text{dist}(q,S) \leq d(p,q) \). Interchange \( p \) and \( q \), we get \( \text{dist}(q,S) - \text{dist}(p,S) \leq d(p,q) \). So we have \( | \text{dist}(p,S) - \text{dist}(q,S) | \leq d(p,q) \) for any \( p, q \in M \).
Given \( \varepsilon > 0 \), take \( \delta = \varepsilon \), then we get if \( d(p,q) < \delta \Rightarrow | \text{dist}(p,S) - \text{dist}(q,S) | < \varepsilon \). Therefore the function \( p \mapsto \text{dist}(p,S) \) is uniformly continuous function of \( p \in M \).

27-(a):
Since \( A_n \) are all non-empty, pick \( a_n \in A_n \). We claim that \( \{ a_n \} \) is a Cauchy sequence. Given \( \varepsilon > 0 \), since \( \text{diam} A_n \to 0 \), there exist \( N \) such that \( \text{diam} A_n < \varepsilon \) for all \( n \geq N \). If \( m, n \geq N \), then we have \( a_m, a_n \in A_N \) since \( A_n \) is nested decreasing sets. Therefore we get \( d(a_m, a_n) \leq \text{diam} A_N < \varepsilon \). This proves that \( \{ a_n \} \) is a Cauchy sequence.
Since \( M \) is complete, there exist \( a \in M \) such that \( a_n \to a \). Next we claim that \( a \in \cap A_n \). We know \( a_k \in A_n \) for all \( k \geq n \) and \( a_k \to a \). Therefore, \( a \) is a limit point of \( A_n \) for all \( n \). Since \( A_n \) is closed, we have \( a \in A_n \) for all \( n \). That is, \( a \in \cap A_n \).
Finally, we will prove that $a$ is the only point in $\cap A_n$. Suppose we have $b \neq a$ and $b \in \cap A_n$, then $b \in A_n$ for all $n$. Since $a \neq b$, we hence $d(a, b) = \alpha > 0$. But $a, b \in A_n$ for all $n$ and $\text{diam} A_n \to 0$, we must have $\alpha \leq \text{diam} A_n$ for all $n$, which is impossible. Therefore, such $b$ does not exist and $a$ is the unique point in $\cap A_n$.

40-(a):
Denote $G = \{(p, y) \in M \times \mathbb{R} : y = fp\}$. Let $(p, y)$ be a limit of $G$, then there exist $(p_n, fp_n) \in G$ such that $(p_n, fp_n) \to (p, y)$. By theorem 21-(d), we have $p_n \to p$ and $fp_n \to y$. Since $f$ is continuous, $fp_n \to fp$. Therefore $y = fp$, and $(p, y) = (p, fp) \in G$. Therefore, $G$ contains all its limits and hence is closed. 

40-(b):
Denote $fM = \{fp : p \in M\}$ the image of $M$ under $f$. Since $f$ is continuous and $M$ is compact, by theorem 39, $fM$ is compact. Also by theorem 31, we know $M \times fM$ is compact. $G \subset M \times fM$ and by (a) it is closed. Therefore, by theorem 35, we know $G$ is compact. 

40-(c):
Suppose $f$ is not continuous at $p \in M$, then there exist $\varepsilon > 0$, such that for all $n \in \mathbb{N}$, we have $p_n \in M$ such that $d(p_n, p) < \frac{1}{n}$ and $|fp_n - fp| > \varepsilon$ for all $n$. Then consider $(p_n, fp_n) \in G$, since $G$ is compact, there exist subsequence $(p_{n_k}, fp_{n_k})$ such that $(p_{n_k}, fp_{n_k}) \to (q, fq) \in G$ for some $q \in M$. Then we have $p_{n_k} \to q$, but $p_{n_k} \to p$ too since $p_n \to p$. So, $q = p$. And $fp_{n_k} \to fp$, but this contradict to the fact that $|fp_n - fp| > \varepsilon$ for all $n$. Therefore, $f$ is continuous on $M$. 

40-(d):
Take $f = \frac{1}{x}$ for $x \neq 0$, and $f0 = 0$. 