Math 5616 – Homework II
Due Friday, February 20

Chapter 4, pages 251–257: 30a, 40
Pre-lim problems, pages 258–266: 3 (Hint: use cor. 40, page 80)

Plus the following extra problems.

X1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)x^n dx = 0$ for all $n$. Follow the steps below to see that $f$ must be identically 0 on $[0, 1]$. First show that $\int_0^1 f(x)P_n(x) dx = 0$ where $P_n(x)$ is any polynomial. Then use the Weierstrass approximation theorem to prove that $\int_0^1 f(x)^2 dx = 0$. Finally use this to conclude that $f(x) = 0$ for all $x \in [0, 1]$.

X2. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous even function, i.e., $f(-x) = f(x)$. Show that there is a sequence of even polynomials $P_n(x)$, $P_n(-x) = P_n(x)$, which converges uniformly to $f$ on $[-1, 1]$. Hint: first consider $[0, 1]$.

X3. Call a vector $X = (x, y) \in \mathbb{R}^2$ resonant if it satisfies an equation of the form $ax + by + c = 0$ where $a, b, c$ are integers, not all zero. Otherwise $X$ is non-resonant. Use Baire theory to show that the set of non-resonant vectors is dense in $\mathbb{R}^2$.

X4. Let $S \subset C^0([0, 1], \mathbb{R})$ be the set of continuous functions which are so “wiggly” that they are not monotonic in any interval $[a, b] \subset [0, 1]$. In this problem you will use Baire theory to see that the typical continuous function is a member of $S$. Recall that being monotonic in $[a, b]$ means that either $f(x) \leq f(y)$ holds for all pairs of points $a \leq x < y \leq b$ or else $f(x) \geq f(y)$ holds for all such pairs.

a. Fix any $a, b$ such that $0 \leq a < b \leq 1$ and let $U(a, b) = \{f \in C^0([0, 1], \mathbb{R}) | f$ is not monotonic on $[a, b]\}$. Show that $U(a, b)$ is an open and dense subset of $C^0([0, 1], \mathbb{R})$.

b. Show that $S$ is a thick subset of $C^0([0, 1], \mathbb{R})$ (hint: consider all intervals with rational endpoints).