Math 5616 – Homework V.ii
Due Monday, April 27

Chapter 5: 54, 56, 57, 61, 62, 63, 64 (first two questions)

Plus the following extra problems.

X1. This problem uses the change of variables formula and Fubini’s theorem to find the $n$-dimensional volume (Riemann measure) of the $n$-dimensional solid ball $B_n(r) = \{ x \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 \leq r^2 \}$. Let $V(B_n(r)) = |B_n(r)|$ denote the $n$-dimensional Riemann measure and let $c_n = V(B_n(1))$ be the measure of the unit ball. We have $V(B_1(r)) = 2r, V(B_2(r)) = \pi r^2$ and $c_1 = 2, c_2 = \pi$.

a. Prove that $V(B_n(r)) = c_n r^n$.
b. Use Fubini’s theorem to show that $c_{n+2} = c_n \int_{B_2(1)} (1 - x_1^2 - x_2^2)^{\frac{n}{2}}$.

Hint: if you fix the first two variables, what are the slices?
c. Evaluate the integral in part b and use the resulting recursion formula to find $c_3, \ldots, c_{10}$.

X2. Let $\alpha = f \, dx + g \, dy + h \, dz$ be a one-form on $\mathbb{R}^3$. Define a corresponding two-form (called the Hodge star of $\alpha$) by $\ast \alpha = \omega = f \, dy \wedge dz + g \, dz \wedge dx + h \, dx \wedge dy$. Similarly we define $\ast \omega = \alpha$ for $\alpha, \omega$ as above.

If $f = f(x, y, z)$ is a zero-form (i.e. a smooth function), then $df$ is one-form, $\ast df$ is a two-form and $d \ast df$ is a three-form. Find a formula for $d \ast df$ involving the second partial derivatives of $f$.

X3. (Maxwell’s equations). In physics $\mathbb{R}^4 = \{(x, y, z, t)\}$ is called space-time. Electric and magnetic fields varying in space and time are conventionally viewed as three-dimensional vector fields $E(x, y, z, t) = (E_1, E_2, E_3)$ and $B(x, y, z, t) = (B_1, B_2, B_3)$. However it turns out that it is advantageous to represent them instead as the following differential forms:

$E = E_1 \, dx + E_2 \, dy + E_3 \, dz \quad B = B_1 \, dy \wedge dz + B_2 \, dz \wedge dx + B_3 \, dx \wedge dy.$

Show that if this convention is adopted, then two of the famous Maxwell equations, namely

$$\frac{\partial B}{\partial t} = -\text{curl} \, E \quad \text{div} \, B = 0$$

are equivalent to the statement that $E \wedge dt + B$ is a closed two-form.