Math 5616 – Homework VI – Solutions

X1. Pull-back practice. Calculate the pullback $F^*\omega$ of the given form under the given map.

b. $\omega = (y_1^2 + y_2^2)dy_1 \wedge dy_2$, $F(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$.
Substitute $y_1 = x_1^2 - x_2^2, y_2 = 2x_1x_2, dy_1 = 2x_1dx_1 - 2x_2dx_2, dy_2 = 2x_2dx_1 + 2x_1dx_2$
to get $F^*\omega = 4(x_1^2 + x_2^2)^3 dx_1 \wedge dx_2$.

c. $\omega = dy_1 \wedge dy_2 + dy_3 \wedge dy_4$, $F(x_1, x_2, x_3, x_4) = (\sqrt{2x_1} \cos x_2, \sqrt{2x_1} \sin x_2, \sqrt{2x_3} \cos x_4, \sqrt{2x_3} \sin x_4)$.
Substitute $y_1 = \sqrt{2x_1} \cos x_2, y_2 = \sqrt{2x_1} \sin x_2$
and $dy_1 = \cos x_2 \, dx_1/\sqrt{2x_1} - \sqrt{2x_1} \sin x_2 \, dx_2, dy_2 = \sin x_2 \, dx_1/\sqrt{2x_1} + \sqrt{2x_1} \cos x_2 \, dx_2$
and wedge to get $dy_1 \wedge dy_2 = dx_1 \wedge dx_2$. The other two coordinates are
similar so $F^*\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, i.e., the original form again.

X2. Integration practice. For each of the forms $\omega$ and maps $F$ from part a, and each of the cells $\phi$ given below, verify (by calculating both sides separately) the pull-back formula

$$\int_{F^*\phi} \omega = \int_{\phi} F^*\omega.$$ 

b. $\phi = (u_1 \cos 2\pi u_2, u_1 \sin 2\pi u_2), (u_1, u_2) \in [0, 1]^2$.

$$\int_{\phi} F^*\omega = \int_{[0,1]^2} 4(x_1^2 + x_2^2)^3 \, dx_1 \wedge dx_2 = \int_{[0,1]^2} 4(\phi_1^2 + \phi_2^2)^3 \, \det D\phi(1,2).$$
The determinant is $2\pi u_1$ and $\phi_1^2 + \phi_2^2 = u_1^2$ so

$$\int_{\phi} F^*\omega = \int_0^1 \int_0^1 8\pi u_1^3 \, du_1 \, du_2 = \pi.$$ 

On the other hand, $F \circ \phi = F \circ (u_1^2 \cos 4\pi u_2, u_1^2 \sin 4\pi u_2)$ so

$$\int_{F \circ \phi} \omega = \int_{F \circ \phi} (u_1^2 + y_2^2) \, dy_1 \wedge dy_2 = \int_{[0,1]^2} u_1^4 \, \det D(F \circ \phi)(1,2).$$
The determinant is $8\pi u_1^3$ so the integral is the same as before.
c. \( \phi = (\cos 2\pi u_1, \sin 2\pi u_1, \cos 2\pi u_2, \sin 2\pi u_2), (u_1, u_2) \in [0, 1]^2 \).

First compute
\[
\int F^* \omega = \int_{\phi} dx_1 \wedge dx_2 + dx_3 \wedge dx_4 = \int_{[0,1]^2} \det D\phi_{(1,2)} + \det D\phi_{(3,4)}.
\]

Since the first two components of \( \phi \) depend only on \( u_1 \) the 2 \times 2 matrix \( D\phi_{(1,2)} \) has a zero second column and so the determinant is 0. Similarly for \( D\phi_{(3,4)} \). So \( \int \phi F^* \omega = 0 \).

On the other hand
\[
\int F^* \phi \omega = \int_{F \circ \phi} dy_1 \wedge dy_2 + dy_3 \wedge dy_4 = \int_{[0,1]^2} \det (F \circ \phi)_{(1,2)} + \det (F \circ \phi)_{(3,4)}.
\]

Now \( (F \circ \phi)_{(1,2)} = (\sqrt{2} \phi_1 \cos \phi_2, \sqrt{2} \phi_1 \sin \phi_2) \) and as above, \( \phi_1, \phi_2 \) depend only on \( u_1 \). So \( \det (F \circ \phi)_{(1,2)} = 0 \) and similarly \( \det (F \circ \phi)_{(3,4)} = 0 \).

So \( \int \alpha = \int \beta = 0 \) since \( \partial \phi = 0 \).

X4. We have \( d\alpha = 2x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_2 = 0 \) so \( \alpha \) is closed. By the Poincaré lemma, \( \alpha \) is exact, i.e., \( \alpha = d\beta \) for some \( \beta \). By Stokes’ theorem
\[
\int \phi \alpha = \int_{\partial \phi} \beta = 0
\]

X5. By Stokes’ theorem and \( d\omega = 0 \)
\[
\int_H d\omega = \int_{\partial H} \omega.
\]

The boundary is
\[
\partial H = -\delta_{1,0} H + \delta_{1,1} H + \delta_{2,0} H - \delta_{2,1} H.
\]
\[
\delta_{1,0} H = H(0, u_2) = H(1, u_2) = \delta_{1,1} H
\]
by definition of a free homotopy of closed curves, whereas \( \delta_{2,0} H = \phi(u_1) \) and \( \delta_{2,1} H = \psi(u_1) \). So the integrals over the first two parts of the boundary cancel and the remaining two parts give
\[
0 = \int \omega - \int \psi.
\]

X6. Intuitively, \( \omega = d\theta \) where \( \theta \) is the polar coordinate angle. The problem is to find a well-defined formula \( f(x, y) \) for the angle. Using the hint and the fact that \( \cos \theta = x/r, \sin \theta = y/r \) gives
\[
f(x, y) = 2 \tan^{-1} \frac{y}{r + x} = 2 \tan^{-1} \frac{y}{\sqrt{x^2 + y^2} + x}
\]
which is smooth on \( V \) and has \( df = \omega \).