Math 8502 — Homework I

due Friday, February 22. Write up any 4 of these 5 problems.

1. Consider a scalar, autonomous ODE \( \dot{x} = f(x), \ x \in \mathbb{R}^1 \), where \( f(x) \) is a polynomial of degree at least 2. Show that there is at least one maximal solution \( x(t) \) which is not defined for all \( t \in \mathbb{R} \).

2. Let \( \phi_t(x), \psi_t(y) \) be two flows on \( \mathbb{R}^n \). They are called linearly conjugate if there is an invertible linear map \( y = Qx \) such that
\[ Q\phi_t(x) = \psi_t(Qx) \quad \text{for all} \ t \in \mathbb{R}, x \in \mathbb{R}^n. \]
They are topologically conjugate if there is a homeomorphism \( y = h(x) \), \( h : \mathbb{R}^n \to \mathbb{R}^n \), such that
\[ h(\phi_t(x)) = \psi_t(h(x)) \quad \text{for all} \ t \in \mathbb{R}, x \in \mathbb{R}^n. \]

a. Let \( A, B \) be two \( n \times n \) real matrices. The corresponding linear flows are given by \( \phi_t(x) = e^{tA}x, \psi_t(y) = e^{tB}y \). Show that they are linearly conjugate if and only if the two matrices \( A, B \) are similar.

b. Show that the linear flows determined by the matrices below are topologically conjugate but not linearly conjugate. Here \( a, b \) are any two positive numbers not both equal to 1. Hint: Try a map, \( h \), of the form
\[ y = (y_1, y_2) = (\text{sgn}(x_1)|x_1|^\alpha, \text{sgn}(x_2)|x_2|^\beta) \]
\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}. \]

c. Same as part b. but for the matrices below, where \( a > 0, b \neq 0 \). Hint: It is possible to find an explicit formula for \( h(x) \). One approach uses the fact that the distance to the origin \( r(t) \) is decreasing for both flows. Let \( t_1(x) \) be the time when \( \phi_t(x) \) crosses the unit circle (find a formula for it) and consider \( h(x) = e^{-t_1(x)B}e^{t_1(x)A}x. \)
\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}. \]

d. Show that the linear flows determined by the matrices below are not topologically conjugate.
\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}. \]

3. Let \( \phi_t(x) \) be a flow on phase space \( \mathcal{D} \). Suppose \( \phi_t(x_0) \) exists for all \( t \geq 0 \). Define the omega limit set to be the set of limit points of the forward orbit:
\[ \omega(x_0) = \{ y \in \mathcal{D} : \exists t_n \to \infty, \phi_{t_n}(x_0) \to y \}. \]
Suppose that there is a compact subset \( K \subset \mathcal{D} \) such that \( \phi_t(x_0) \in K \) for all \( t \geq 0 \). Show that \( \omega(x_0) \) is a non-empty, compact subset of \( K \). Also show that \( \omega(x_0) \) is an invariant set and that orbits in \( \omega(x_0) \) exist for all \( t \in \mathbb{R} \), i.e., show that if \( y \in \omega(x_0) \) then for all \( t \in \mathbb{R} \), \( \phi_t(y) \) exists and \( \phi_t(y) \in \omega(x_0). \)
4. The Lorenz Equation. Consider the following ODE in $\mathbb{R}^3$: 
\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{align*}
\]
where $\sigma > 0$, $b > 0$, $r > 0$ are parameters.

a. Find all the equilibrium points. For which values of the parameters are they non-degenerate? For which values of the parameters are they hyperbolic and what are the dimensions of the stable and unstable manifolds?

b. Show that the $z$-axis is an invariant set which is contained in the stable manifold of the origin: $W^s(0)$.

c. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map $L(x,y,z) = (-x,-y,z)$. Geometrically, $L$ is rotation around the $z$-axis by 180 degrees. Show that $L$ is a symmetry of the flow of the Lorenz equation, i.e., if $(x(t), y(t), z(t))$ is a solution, so is $L(x(t), y(t), z(t))$. Show that $L$ leaves the stable and unstable manifolds $W^s(0)$ and $W^u(0)$ invariant.

d. Show that if $r < 1$ then the Lorenz flow is gradient-like with respect to the function $g(x,y,z) = \frac{1}{2}(x^2/\sigma + y^2 + z^2)$, i.e., this function is strictly decreasing except at the restpoints. Use this to show that, in this case, $W^s(0) = \mathbb{R}^3$, i.e., every solution converges to 0.

5. (Linearized Hamiltonian Systems) Let $q \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ and let $z = (q, p) \in \mathbb{R}^{2n}$. Consider a Hamiltonian system of ODEs:
\[
\begin{align*}
\dot{q} &= H_p \\
\dot{p} &= -H_q \\
\dot{z} &= J\nabla H
\end{align*}
\]
or
\[
\dot{z} = J\nabla H, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]

a. A $2n \times 2n$ matrix, $A$, is called Hamiltonian if $JA$ is symmetric. If $\bar{z}$ is an equilibrium point of a Hamiltonian system, show that the linearized ODE is of the form $\dot{v} = Av$ where $A$ is a Hamiltonian matrix.

b. $B$ is called symplectic if $B^TJB = J$ where $B^T$ is the transpose of $B$. Show that $A$ is Hamiltonian if and only if $B = e^{tA}$ is symplectic for all $t$. Hint: differentiate the expression $(e^{tA})^TJ e^{tA}$.

c. If $A$ Hamiltonian matrix, show that the characteristic polynomial $p(\lambda) = |A - \lambda I|$ is an even function, i.e., $p(-\lambda) = p(\lambda)$. If $B$ is symplectic show that $\lambda^{2n}p(1/\lambda) = p(\lambda)$. Hint: Start by multiplying $|A - \lambda I|$ on the left by $|J| = 1$. 