1. (Smooth linearization in $\mathbb{R}^1$). Let $x \in \mathbb{R}^1$ and consider an analytic ODE (* $\dot{x} = F(x)$) with a hyperbolic equilibrium point at the origin. Then we can write $F(x) = \lambda(x + x^2 f(x))$ with $\lambda \neq 0$ and $f(x)$ analytic. Show that (*) can be linearized by an analytic local diffeomorphism $x = k(y) = y(1 + g(y))$ where $g(y)$ is an analytic function with $g(0) = 0$. Similarly, if $F$ is $C^\infty$, it can be linearized by a $C^\infty$ $k$. Hint: Show that $k$ linearizes (*) provided $g(y)$ solves a certain ODE.

2. (Normal form at a center). Consider the ODE in $\mathbb{R}^2$

$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x^2 + xy \\ 0 \end{bmatrix}.$

a. Make the complex, linear coordinate change $z = x + iy, w = x - iy$ to get a system of the form

$\begin{bmatrix} \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} + \begin{bmatrix} f_2(z,w) \\ g_2(z,w) \end{bmatrix}$

where $f_2, g_2$ are quadratic polynomials.

b. Find a further coordinate change of the form $\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix} + \begin{bmatrix} f_3(z,w) \\ g_3(z,w) \end{bmatrix}$ to eliminate the quadratic terms in the ODE. You will get an ODE of the form

$\begin{bmatrix} \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} + \begin{bmatrix} f_3(z,w) \\ g_3(z,w) \end{bmatrix} + \ldots.$

Compute the third degree terms $f_3, g_3$. Hint: Although the coordinate change is quadratic, its inverse is not. You will need to do some work to find the correct third degree terms.

c. The theory tells us that there exists another coordinate change eliminating the non-resonant monomials of $f_3, g_3$, leaving the resonant terms unchanged (fortunately, you don’t actually have to find this coordinate change). Keeping only the resonant terms of $f_3, g_3$ write down this normal form. Then convert back to $(x, y)$ coordinates and use the resulting real normal form to discuss the stability of the equilibrium point at the origin.

3. Consider the following ODE in $\mathbb{R}^2$ with two parameters $(\sigma, \gamma)$:

$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x - \sigma y - x(x^2 + y^2) \\ \sigma x + y - y(x^2 + y^2) - \gamma \end{bmatrix}.$

a. Find the “bifurcation curve”, $\mathcal{B}$, in the $(\sigma, \gamma)$ such that $(\sigma, \gamma) \in \mathcal{B}$ if and only if the ODE has a degenerate equilibrium point. In each connected component of the complement of $\mathcal{B}$ determine how many equilibrium points the system has. Hint: Use a computer to eliminate $(x, y)$ from the equations defining degenerate equilibrium points. You should get a polynomial equation in $(\sigma, \gamma)$ whose solutions you can plot.

b. Consider the one parameter family of ODE’s obtained from the family above by setting $\sigma = 0$. Show that there is a saddle-node bifurcation at a certain parameter value $\gamma_0 > 0$. 

4. The next two problems deal with an example due to Hartman of an ODE with a hyperbolic equilibrium point which cannot be locally linearized by a $C^1$ diffeomorphism. Here is the ODE and it’s linearization:

\[
\begin{align*}
(\ast) \quad & \dot{x} = Ax + \begin{bmatrix} 0 \\ \epsilon x_1 x_3 \\ 0 \end{bmatrix}, \quad \dot{y} = Ay, \\
& A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha - \gamma & 0 \\ 0 & 0 & -\gamma \end{bmatrix}
\end{align*}
\]

where $x, y \in \mathbb{R}^3$, $\alpha > \gamma > 0$ and $\epsilon > 0$.

a. Using elementary methods, find the flow $\phi_t(x_1, x_2, x_3)$ of system (\ast).

b. For the linearized system, the three coordinate axes and the three coordinate planes are obviously invariant under the flow. Some of these six subspaces are actually still invariant for the nonlinear system (\ast). Which ones? Which of these invariant subspaces constitute the stable and unstable manifolds of the origin for (\ast)?

c. Show that the mapping $x = k(y) = (y_1, y_2 - \epsilon y_1 y_3 \log |y_3|/\gamma, y_3)$ is a topological conjugacy between the linear flow $\psi_t(y)$ and $\phi_t(x)$. Show that $k$ is not differentiable.

d. From part b., the $(y_1, y_3)$-plane, call it $P$, is invariant for the linearized system but not for nonlinear system (\ast). Show that it’s image $S = k(P)$ is an invariant two-dimensional surface containing the $x_1$ and $x_3$ axes. Show that along the $x_1$-axis, $S$ is tangent to the $(x_1, x_2)$-plane.

5. Continuing the discussion of Hartman’s example from problem 4. Assume for the sake of contradiction that there is a $C^1$ diffeomorphism $x = k(y)$, $k : \mathcal{V} \to \mathcal{U}$, which conjugates the linear and nonlinear flows. Here $\mathcal{U}, \mathcal{V}$ are neighborhoods of the origin.

a. Let $P$ be the $(y_1, y_3)$-plane and $S = k(P)$ as in problem 4 d. Show that part of $S$ in $\mathcal{U}$ is a $C^1$ locally invariant surface containing the parts of the $x_1$ and $x_3$ axes in the neighborhood $\mathcal{U}$. Moreover, $S$ meets the $(x_2, x_3)$-plane transversely along the $x_3$-axis and meets the $(x_1, x_2)$-plane transversely along the $x_1$-axis (rather than tangentially as for the non-smooth conjugacy in problem 4).

b. Now show that any locally invariant surface $S$ meeting the $(x_2, x_3)$-plane transversely along the $x_3$-axis must meet the $(x_1, x_2)$-plane tangentially along the $x_1$-axis. To do this, take a small curve in $S$ of the form $x_3 = c$ and follow it through a neighborhood of the origin to a plane $x_1 = d$. If the initial curve is transverse to the $(x_1, x_2)$-plane, show that the final curve is tangent to the $(x_1, x_2)$-plane. This shows that surface $S$ of the required form does not exist.