Math 8502 — Homework III

due Friday, May 6 (last day of class)

1. Hill’s problem is a simplified version of the restricted three-body problem given by the following system of differential equations

\[
\begin{align*}
\dot{x} &= u \\
\dot{u} &= 2v + V_x \\
\dot{y} &= v \\
\dot{v} &= -2u + V_y
\end{align*}
\]

where \( V(x,y) = 3x^2 + \frac{1}{\sqrt{x^2 + y^2}} \).

a. Show that \( H = \frac{1}{2}(u^2 + v^2) - V(x,y) \) is a first integral.

b. Find the equilibrium points and their eigenvalues. Show that there are families of periodic orbits nearby and use the eigenvectors of the linearized ODE to approximate how these orbits look in the \((x,y)\)-plane.

c. Consider the related system:

\[
\begin{align*}
\dot{X} &= U \\
\dot{U} &= 2\epsilon^3 V + W_X \\
\dot{Y} &= V \\
\dot{V} &= -2\epsilon^3 U + W_Y
\end{align*}
\]

where \( W(X,Y) = 3\epsilon^6 X^2 + \frac{1}{\sqrt{X^2 + Y^2}} \) and \( \epsilon \) is a small parameter.

If \((X(t),Y(t))\) is a periodic solution of this system for \( \epsilon \neq 0 \), show that

\[
\left( x(t), y(t) \right) = \epsilon^2 \left( X(t/\epsilon^3), Y(t/\epsilon^3) \right)
\]

is a periodic solution of Hill’s equation (very close to the singularity at the origin).

d. When \( \epsilon = 0 \), the system in part c reduces to the Kepler problem. This has many circular periodic solutions which we would like to continue to \( \epsilon \neq 0 \). Unfortunately, they are all degenerate. Consider instead the system in part c with the \( O(\epsilon^6) \) terms dropped but keeping the \( O(\epsilon^3) \) terms. Show that this system has circular periodic solutions with multipliers of the form \( 1, 1, 1 \pm \epsilon \epsilon^3 + O(\epsilon^6) \) where \( \epsilon \neq 0 \). So they are non-degenerate when \( \epsilon \neq 0 \).

e. (Optional). To prove that the periodic orbits in part d can be continued to periodic solutions of the system in part c (i.e., with the \( O(\epsilon^6) \) terms included again) requires a variation on the usual continuation theorem. After choosing a Poincaré section inside a manifold of constant energy, the question reduces to finding fixed points of a smooth family of maps \( F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) such that \( F(z,\epsilon) = \epsilon^3 F_1(z) + \epsilon^6 F_2(z,\epsilon) \) where \( F_1(0) = 0 \) and \( DF_1(0) \) has eigenvalues \( \pm \epsilon \neq 0 \). Prove that for such a family of maps, the fixed point at the origin can be continued to \( \epsilon \neq 0 \) sufficiently small.

2. (Shift Maps). Let \( l \geq 2 \) and define \( \Sigma_l \) to be the set of all bi-infinite sequences on \( l \) symbols, i.e., sequences \( \epsilon = \ldots \epsilon_{-\infty} \epsilon_0 \epsilon_1 \ldots \) with \( \epsilon_k = 0, 1, \ldots, l - 1 \). Let \( \sigma : \Sigma_l \to \Sigma_l \) be the shift map, \( \sigma(\epsilon)_k = \epsilon_{k+1} \). Finally, define the distance between two sequences to be

\[
d(\epsilon, \epsilon') = \sum_{k=-\infty}^{\infty} \frac{|\epsilon_k - \epsilon'_k|}{l|k|}.
\]

a. Show that \( \Sigma_l \) is a compact metric space. Hint: for compactness it suffices to show that every sequence in \( \Sigma_l \) has a convergent subsequence.

b. Show that the shift map is a homeomorphism.
c. We showed in class that when $l = 2$ there is a sequence whose forward orbit under $\sigma$ is dense in $\Sigma_2$. Let $D \subset \Sigma_l$ be the set of all sequences with dense forward orbits. Show that $D$ is a residual set, i.e., it is an intersection of countably many sets each of which is open and dense in $\Sigma_l$. (Since $\Sigma_l$ is a “Baire space”, it follows that $D$ is dense in $\Sigma_l$; see a book on topology for more about Baire category). Hint: Consider the set of all sequences $\epsilon$ containing a given finite sequence as a subsequence of their forward halves.

d. Let $\Lambda \subset \mathbb{R}^2$ be the invariant set of orbits of the Smale horseshoe map which remain in the unit square. We constructed a 1-1 correspondence $h : \Sigma_2 \to \Lambda$ by using itineraries with respect to the horizontal boxes $H_0, H_1$. Show that $h$ is a homeomorphism.

3. (Subshifts of finite type). Let $G$ be any directed graph with vertices labelled $0, 1, \ldots, l - 1$. In other words, $G$ consists of the vertices together with arrows connecting some pairs of vertices. Associated with $G$ is a transition matrix $M$ such that $M_{ij} = 1$ if there is a directed edge $j \to i$ and $M_{ij} = 0$ otherwise. Also, there is an associated subset $\Sigma' \subset \Sigma_l$ consisting of all sequences $\epsilon \in \Sigma_l$ such that for every $k$, the vertices with labels $\epsilon_k$ and $\epsilon_{k+1}$ are connected by a directed edge in $G$ (thus $G$ is a graphical representation of some rules about which symbols may be adjacent in $\epsilon$).

a. Show that $\Sigma'$ is a compact subset of $\Sigma_l$ which is invariant under the shift map.

b. Consider the powers $M^n$ of the transition matrix $M$. Prove by induction that $M^n_{ij}$ is the number of distinct paths of length $n$ connecting vertex $j$ to vertex $i$. Let $T(n)$ be the number of periodic sequences in $\Sigma'$ of (not necessarily minimal) period $n$. Show that $T(n) = \text{tr} M^n$.

c. Consider the subshift $\Sigma' \subset \Sigma_2$ with transition matrix $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Prove that the associated subshift is chaotic, i.e., periodic points are dense, sensitive dependence on initial conditions and existence of a dense orbit. Finally, find the nicest explicit formula that you can for the number $P(n)$ of periodic points of $\Sigma'$ with minimal period $n$.

4. Apply the Melnikov integral method to show that the periodically forced Duffing equation (written below) has a periodic orbit with a transverse homoclinic orbit for $\epsilon$ sufficiently small. The unperturbed problem with $\epsilon = 0$ is Hamiltonian. You should sketch the phase portrait, find the unperturbed homoclinic orbit and then compute the Melnikov integral. (You may assume that the theory applies in this case even though the perturbation does not vanish along the unperturbed periodic orbit).

\[ \ddot{q} - q + q^3 = \epsilon \sin \omega t. \]