Math 8520 – Homework II
Due Wednesday, November 9
Write up solutions for any 4 of the following 5 problems.

1. Suppose a Lie group $G$ acts on a manifold $M$ and the a Lagrangian $L : TM \to \mathbb{R}$ is symmetric under the corresponding action on $TM$. The action is given by a smooth map $\psi : G \times M \to M$ and the image of a point $\gamma \in M$ under $G$ can be written in several ways:

$$g \cdot \gamma = \psi(g, \gamma) = \psi_g(\gamma).$$

We have $\psi_g = \psi_h \circ \psi_h$ for $g, h \in G$. The action on $TM$ is

$$g \cdot (\gamma, v) = T\psi_g(\gamma, v) = (\psi_g(\gamma), D\psi_g(\gamma)v).$$

(a) For $a \in \mathfrak{g}$ we have the symmetry vectorfield $\chi_a(\gamma) = \frac{d}{ds} e^{sa} \cdot \gamma |_{s=0}$. Show that these vectorfields satisfy:

$$\chi_a(g \cdot \gamma) = D\psi_g(\gamma) \chi_a(g^{-1}ag)(\gamma).$$

(b) Let $J : TM \to \mathfrak{g}^*$ be the momentum map, given by

$$J(\gamma, v)(a) = \frac{\partial L}{\partial v}(\gamma, v) \cdot \chi_a(\gamma).$$

Show that

$$J(g \cdot (\gamma, v))(a) = J(\gamma, v)(g^{-1}ag)$$

for every $(\gamma, v) \in TM, g \in G, a \in \mathfrak{g}$.

2. A $2m \times 2m$ matrix $A$ is called symplectic if $A^TJA = J$ where $J$ is the block matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$ 

The symplectic matrices form a matrix group denoted $Sp(2m)$ called the symplectic group. Show that $Sp(2m)$ is a Lie group of dimension $m(2m+1)$ and describe its Lie algebra $sp(2m) = T_I Sp(2m)$.

3. (4D Angular Momentum) Let $q \in \mathbb{R}^4$ and consider a Lagrangian system with Lagrangian $L(q, v) = \frac{1}{2}|v|^2 - U(q)$, where $U(q) = f(|q|)$ is a function of the radius $|q|$ (the central force problem in $\mathbb{R}^4$). $SO(4)$ acts as a symmetry group of this Lagrangian system and so there is a momentum map $J(q, v)$ with values in $so(4)^*$. Recall that $so(4)$ is the set of all antisymmetric $4 \times 4$ matrices. Such a matrix also represents a two-form via the correspondence

$$\tilde{\alpha} = \begin{bmatrix} 0 & -\alpha_{12} & -\alpha_{13} & -\alpha_{14} \\ \alpha_{12} & 0 & -\alpha_{23} & -\alpha_{24} \\ \alpha_{13} & \alpha_{23} & 0 & -\alpha_{34} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & 0 \end{bmatrix} \leftrightarrow \alpha = \alpha_{12}dx_1 \wedge dx_2 + \ldots + \alpha_{34}dx_3 \wedge dx_4.$$

We can identify $so(4)^*$ with $so(4)$ as follows: $\tilde{\alpha}$ represents the element of $so(4)^*$ such that

$$\tilde{\alpha}(\beta) = \alpha_{12} \beta_{12} + \ldots + \alpha_{34} \beta_{34}.$$ 

(a) With these conventions, the momentum map should associate to every $(q, v) \in T\mathbb{R}^4$, a two-form $J(q, v) = \alpha_{12}dx_1 \wedge dx_2 + \ldots + \alpha_{34}dx_3 \wedge dx_4$. Show that if we view the vectors $q$ and $v$ as one-forms $q = q_1dx_1 + \ldots, v = v_1dx_1 + \ldots$, then $J(q, v) = q \wedge v$.

(b) A two-form which is the wedge product of two one-forms is called reducible (for example, the two-form $J(q, v)$ from part a). Not every two-form is reducible so not every element of $so(4)^*$ is in the image of the momentum map. To see this show that if $\alpha = \alpha_{12}dx_1 \wedge dx_2 + \ldots$ is reducible, then

$$\alpha_{12} \alpha_{34} - \alpha_{13} \alpha_{24} + \alpha_{14} \alpha_{23} = 0.$$
Essentially, this equation shows that only five of the six angular momentum integrals $\alpha_{ij}$ are independent.

(c) Show that if $J(q, v) \neq 0$ then using the action of $SO(4)$ one can assume without loss of generality that $J(q, v) = \alpha_{12} dx_1 \wedge dx_2$ (all other $\alpha_{ij} = 0$). Show that the corresponding level manifold $M = \{(q, v) : J(q, v) = \alpha\} = \{q_3 = q_4 = v_3 = v_4 = 0, q_1 v_2 - q_2 v_1 = \alpha_{12}\}$. (This reduces the whole problem to a central force problem in $\mathbb{R}^3$).

4. (Topology of $SO(3)$) The configuration manifold of the rigid body in $\mathbb{R}^3$ is the rotation group $SO(3)$. This problem explores some topological properties of this three-dimensional manifold.

(a) Show that $SO(3)$ is a compact subset of the set of all $3 \times 3$ matrices (the latter space can be identified with $\mathbb{R}^9$).

(b) Show that $SO(3)$ is path-connected. Hint: The columns of $A \in SO(3)$ form a right-handed orthonormal frame. Describe a way to continuously deform any such frame to the standard basis.

(c) Show that any matrix $A \in SO(3)$ has $\lambda = 1$ as an eigenvalue. Use this to show that every $A \in SO(3)$ is the exponential of some antisymmetric matrix, $\hat{\alpha}$, i.e., $A = e^{\hat{\alpha}}$ for some $\hat{\alpha} \in so(3)$. Hint: Rotate around the $\lambda = 1$ eigenvector.

(d) Let $T_1 S^2$ be the unit tangent bundle of the standard two-sphere in $\mathbb{R}^3$, i.e., the set of all unit vectors tangent to the sphere. Find a diffeomorphism between $T_1 S^2$ and $SO(3)$. Hint: Think of the first column of an orthogonal matrix as a point of $S^2$.

5. (Quaternions and Rotations.) A quaternion is an expression of the form $q = x + iy + jz + kw$. $q$ can be viewed as a point $\mathbb{R}^4$ and quaternions are added component-wise. But they can also be multiplied like complex numbers using the rules

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i^2 = j^2 = k^2 = -1 \quad ij = -ji = k \quad ki = -ik = j \quad jk = -kj = i. $$

This multiplication law is associative and distributes over addition but is not commutative. Define the norm of a quaternion by $|q|^2 = x^2 + y^2 + z^2 + w^2$. Then the unit quaternions constitute $S^3$.

(a) Show that $|q_1 q_2| = |q_1||q_2|$.

(b) Fix a unit quaternion $q$ and define left and right multiplication maps by $L_q(r) = q \cdot r$ and $R_q(r) = r \cdot q$ where the $\cdot$ denotes multiplication of quaternions. Show that $L_q$ and $R_q$ are linear maps and that these linear maps are in $SO(4)$. Hint: use part a to see that they are orthogonal.

(c) Let $\Phi_q = L_q^{-1} R_q$. Show that $\Phi_q$ maps the $x$-axis to itself. It follows that it also maps the three-dimensional $(y, z, w)$ coordinate space to itself. Let $A_q$ denote the restriction of $\Phi_q$ to $\mathbb{R}^3 = \{(y, z, w)\}$. Show that $A_q \in SO(3)$. Find the matrix which represents $A_q$ with respect to the coordinates $(y, z, w)$. (The columns should be Hopf maps!)

(d) Show that the map $A : S^3 \to SO(3)$ given by $A(q) = A_q$ satisfies $A(q_1 q_2) = A(q_2)A(q_1)$ and that $A(q_1) = A(q_2)$ if and only if $q_2 = -q_1$ (Hint: consider $\Phi_q$ instead). It is a general fact from topology (invariance of domain theorem) that a continuous, locally injective map of manifolds of the same dimension always takes open sets to open sets. Assuming this, show that $A$ is surjective. (This show that $SO(3)$ can be identified with the real projective space $RP(3)$ which can be defined by identifying antipodal point of $S^3$. Problem 4d then shows that the unit tangent bundle of $S^2$ is also topologically $RP(3)$.)