Math 8520 – Homework III

Hand in any four of the following five problems.
Due Wednesday, December 14

1. (The Levi-Civita Transformation.) Let $\phi : \mathbb{R}^2 \setminus 0 \rightarrow \mathbb{R}^2 \setminus 0$ be the complex squaring function, given in Cartesian coordinates by $(q_1, q_2) = (Q_2^2 - Q_1^2, 2Q_1Q_2)$.
   (a) What is the induced transformation of cotangent bundles $\Phi : T^* (\mathbb{R}^2 \setminus 0) \rightarrow T^* (\mathbb{R}^2 \setminus 0)$?
   (b) Let $H(q, p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$ be the Hamiltonian of the Kepler problem. Find the new Hamiltonian $K(Q, P)$ and write down Hamilton’s equations.
   (c) Fix an energy $H(q, p) = K(Q, P) = h < 0$. Show that if solutions of Hamilton’s equations are parametrized by a new time variable, $s$, such that $Q^i(s) = 4|Q|^2 \dot{Q}^i(t)$, then Hamilton’s equations restricted to the given energy level become the equations for a two-dimensional harmonic oscillator with equal frequencies.

2. (Hamiltonian Systems on Symplectic Manifolds) Here are some examples of Hamiltonian systems on symplectic manifolds other than the canonical $T^* \mathbb{R}^m$.
   (a) The motion of point vortices (little whirlpools) in a two-dimensional fluid is described by an interesting Hamiltonian system which does not arise from a Newtonian force law. Each vortex has a position $(x_i, y_i) \in \mathbb{R}^2$ and a vorticity $\kappa_i \neq 0, i = 1, \ldots, n$. The differential equations describing the motion are
     \[ \dot{x}_i = \sum_{j \neq i} \frac{\kappa_j (y_i - y_j)}{r_{ij}^2} \]
     \[ \dot{y}_i = -\sum_{j \neq i} \frac{\kappa_j (x_i - x_j)}{r_{ij}^2} \]
     where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. Note that this is a first-order ODE for the positions of the vortices.
     Introduce a symplectic structure, $\omega$, on $\mathbb{R}^{2n}$ where $\omega = \sum_{i=1}^{n} \kappa_i dx_i \wedge dy_i$. Show that using this symplectic structure, the differential equations above are Hamilton’s equations for the Hamiltonian
     \[ H = \frac{1}{2} \sum_{i<j} \ln r_{ij}^2 \]
     where the sum is over all pairs of indices with $1 \leq i < j \leq n$.
   (b) The differential equation
     \[ \dot{M} = M \times (A^{-1}M) \]
     where $\times$ is the cross-product arises in the theory of the motion of a free rigid body with a fixed point. The vector $M \in \mathbb{R}^3$ is the “angular momentum vector in body coordinates”. It has constant length $|M| = m_0 > 0$ so it moves on a sphere.
     Define a symplectic structure on the two-sphere $N = \{ M \in \mathbb{R}^3 : |M| = m_0 \}$ by the formula $\omega(M)(V, W) = \frac{1}{m_0} \det(M, V, W)$. Here $V, W \in T_M N$. This is proportional to the usual area element on the sphere. Show that with respect to this symplectic structure, the differential equations above are Hamilton’s equations for the Hamiltonian $H(M) = \frac{1}{2} M^T A^{-1} M$. Hint: You just have to show that the interior product of the vectorfield in the differential equation with $\omega$ agrees with $-dH$ for vectors tangent to the sphere.
3. (Poisson brackets) On a symplectic manifold we defined the Poisson bracket of functions by

\[ \{f, g\} = -\omega(X_f, X_g) \]

where \( X_f, X_g \) are the Hamiltonian vector fields determined by \( f, g \). We saw that \( \{f, H\} = df(X_H) = \dot{f} \)

where \( \dot{f} \) denotes the time derivative of \( f \) along the Hamiltonian flow of \( H \). In \( \mathbb{R}^{2m} = T^*\mathbb{R}^m \) with the canonical symplectic structure, we saw that

\[ \{f, g\} = \sum_{i=1}^{m} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \]

(a) Give a direct proof of the Jacobi identity, \( \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \), for the Poisson bracket in \( \mathbb{R}^{2m} \).

(b) Use the Jacobi identity to show that if \( f \) and \( g \) are two integrals of a Hamiltonian system, then \( \{f, g\} \) is another integral, i.e., the set of functions which are integrals of a given Hamiltonian system is closed under the Poisson bracket.

(c) Consider a system of \( N \) bodies in \( \mathbb{R}^3 \) moving under a general conservative force law. Let \( q_i \in \mathbb{R}^3, p_i \in \mathbb{R}^{3N} \) be their positions and momenta, \( i = 1, \ldots, N \). The motion will be described by a Hamiltonian system on \( T^*\mathbb{R}^{3N} \) with some Hamiltonian \( H(q_1, \ldots, q_N, p_1, \ldots, p_N) \). The angular momentum \( M = (M_1, M_2, M_3) \in \mathbb{R}^3 \) is given by \( M = \sum_{i=1}^{N} q_i \times p_i^T \) where \( \times \) is the cross product. Use part (b) to show that if two of the three components of \( M \) are integrals of such a Hamiltonian system, then so is the third. Also, show that the function \( |M|^2 = M_1^2 + M_2^2 + M_3^2 \) satisfies \( \{|M|^2, M_i\} = 0, i = 1, 2, 3 \).

4. (Symplectic orthogonal complements) Let \( (\mathcal{V}, \omega) \) be a symplectic vectorspace of dimension \( 2m \) and let \( \mathcal{W} \subset \mathcal{V} \) be a subspace. The \textit{symplectic orthogonal complement} is

\[ \mathcal{W}^\perp = \{V \in \mathcal{V} : \omega(V, W) = 0 \text{ for all } W \in \mathcal{W}\}. \]

\( \mathcal{W} \) is \textit{symplectic} if \( \mathcal{W} \cap \mathcal{W}^\perp = \{0\} \), \textit{Lagrangian} if \( \mathcal{W} = \mathcal{W}^\perp \), \textit{isotropic} if \( \mathcal{W} \subset \mathcal{W}^\perp \) and \textit{coisotropic} if \( \mathcal{W}^\perp \subset \mathcal{W} \).

(a) Prove that \( \dim \mathcal{W} + \dim \mathcal{W}^\perp = \dim \mathcal{V} \), that \( \mathcal{W}^\perp \perp \mathcal{W} \) and that if \( \mathcal{W}_1 \subset \mathcal{W}_2 \) then \( \mathcal{W}_2^\perp \subset \mathcal{W}_1^\perp \).

(b) Show that \( \mathcal{W} \) is symplectic if and only if the restriction of \( \omega \) to \( \mathcal{W} \) is non-degenerate, i.e., if and only if \( (\mathcal{W}, \omega|_{\mathcal{W}}) \) is itself a symplectic vectorspace. Conclude that \( \dim \mathcal{W} \) is even and that \( \mathcal{W}^\perp \) is also symplectic.

(c) Show that \( \mathcal{W} \) is isotropic if and only if \( \omega|_{\mathcal{W}} = 0 \). Show that a Lagrangian subspace is isotropic and has \( \dim \mathcal{W} = m \). Show that if \( \mathcal{W} \) is isotropic but not Lagrangian then \( \dim \mathcal{W} < m \) and there is a Lagrangian subspace \( \mathcal{W}' \) with \( \mathcal{W} \subset \mathcal{W}' \).

(d) Suppose \( \mathcal{W} \) is coisotropic and let \( K = \ker(\omega|_{\mathcal{W}}) = \mathcal{W} \cap \mathcal{W}^\perp \). The quotient vector space \( \mathcal{W}/K \) is, by definition, the set of equivalence classes \( [W] \) of vectors in \( \mathcal{W} \) where \( \mathcal{W} \sim \mathcal{W}' \) if \( \mathcal{W}' - \mathcal{W} \in K \). Show that \( \omega|_{\mathcal{W}} \) induces a well-defined symplectic form \( \omega_0 \) on \( \mathcal{W}/K \) by the formula \( \omega_0([V], [W]) = \omega(V, W) \). Hint: In other words, show that \( \omega_0 \) is independent of the choice of representatives of the equivalence classes and that it is non-degenerate.
5. (Some matrix theory). Recall that a $2m \times 2m$ matrix is orthogonal if $A^tA = I$ and symplectic if $A^tJA = J$ where $J$ is the block matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$ 

If we identify $\mathbb{R}^{2m}$ with $C^m$ by associating $(q, p) \in \mathbb{R}^{2m}$ with $z = q + ip \in C^m$ then the matrix $J$ represents the complex-linear map taking $z$ to $iz$. Then it is natural to call a real $2m \times 2m$ matrix complex if it commutes with $J$: $AJ = JA$. Finally, a matrix which is both complex and orthogonal is called unitary. When thinking about these conditions it is sometimes convenient to think of $A$ as consisting of four $m \times m$ blocks:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

In a previous assignment, you showed that the symplectic matrices form a group (called $Sp(2m)$). The orthogonal, complex, and unitary matrices form groups (called $O(2m)$, $GL(m, C)$ and $U(m)$ respectively) but you don’t have to show this here.

(a) Show that the transpose of a symplectic matrix is symplectic.

(b) Show that $A \in GL(n, C)$ if and only if $a = d$ and $b = -c$ and that in this case, multiplying $(q, p)$ by $A$ is equivalent to multiplying $z$ by the complex $m \times m$ matrix $a + ic$.

(c) Show that $U(n) = O(2n) \cap GL(n, C) = O(2n) \cap Sp(2n) = Sp(2n) \cap GL(n, C)$.

(d) Write $A = [C_1, \ldots, C_n, D_1, \ldots, D_n]$ where $C_i$ and $D_i$ are the columns. Show that $A$ is symplectic if and only if

$$\omega(D_i, C_i) = -\omega(C_i, D_i) = 1 \quad \omega(C_i, C_j) = \omega(C_i, D_j) = \omega(D_i, D_j) = 0 \quad i \neq j$$

where $\omega(v, w) = v^tJw$ is the canonical symplectic 2-form (in this case we say the columns form a symplectic basis).

(e) Let $A$ be a symplectic matrix. Prove that $A$ and $A^{-1}$ have the same characteristic polynomial, i.e., $P(\lambda) = |A - \lambda I| = |A^{-1} - \lambda I|$. Deduce that $P(\lambda) = \pm \lambda^{2m}P(1/\lambda)$. From this, show that if $\lambda$ is an eigenvalue of $A$ of multiplicity $k$ then $1/\lambda$ is also an eigenvalue of $A$ of multiplicity $k$. Hint: To get started, consider $|A^tJ - \lambda J|$.

(f) Let $A$ be symplectic and suppose $v, w$ are eigenvectors of $A$ with eigenvalues $\lambda, \mu$ respectively. Show that if $\lambda \mu \neq 1$ then $\omega(v, w) = 0$. Hint: Consider $\omega(Av, Aw)$. 