STURM'S ALGORITHM AND ISOLATING BLOCKS

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Abstract. A topological existence proof for certain solutions of the Newtonian three-body problem is based on the construction of *isolating blocks* for the flow on an integral manifold. An isolating block is a submanifold whose boundary satisfies a convexity condition with respect to the three-body flow. Verifying this convexity condition can be reduced to the problem of checking the sign of a very complicated function of one variable. This can be done numerically, but the goal of this paper is to show that Sturm's algorithm can be used to provide rigorous verification in some cases.

1. Introduction

An *isolating block* for a flow is a subset of the phase space whose boundary is a codimension-one submanifold which is "convex to the flow" in the sense that any orbit meeting the the boundary manifold tangentially moves outside the block in both forward and backward time. This idea was developed by Conley and Easton [3]. It can be used to provide non-perturbative existence proofs for invariant sets in the flow. Once the convexity condition has been verified, one can sometimes use the topological structure of the block and its boundary to show that the interior of the block contains a nonempty invariant set. Moreover, the invariant set may have stable and unstable sets consisting of orbits which enter the block and are captured by the invariant set.

In [2] Conley constructed an isolating block around the well-known collinear Lagrange point of the restricted three-body problem. Recall that the restricted three-body problem describes the motion of a negligibly small mass under the gravitational influence of two larger masses which are assumed to move on circular orbtis of the two-body problem. In rotating coordinates such that the two primary masses remain on the x-axis, the problem admits five equilibrium points, one of which lies on the x-axis between the primaries. If the third mass is placed at this point with zero velocity (in the rotating frame) it remains there for all time. Thus the three masses are in a permanent state of eclipse.

This relative equilibrium motion takes place on a certain fixed-energy manifold of the problem. Conley's goal was to study the nearby energy levels. For small perturbations away from the equilibrium, he constructed simple isolating blocks near the Lagrange points. The construction makes use of the projection of the energy manifold onto the configuration space. Let $(x, y) \in \mathbf{R}^2$ denote the position vector of the small mass in the rotating system and let $(u, v) = (\dot{x}, \dot{y})$ be its velocity vector. The phase space is $\mathbf{R}^4 = \{(x, y, u, v)\}$. Suppose the two large masses are normalized to be $m_1 = 1 - \mu$ and $m_2 = \mu$ and are located at constant positions

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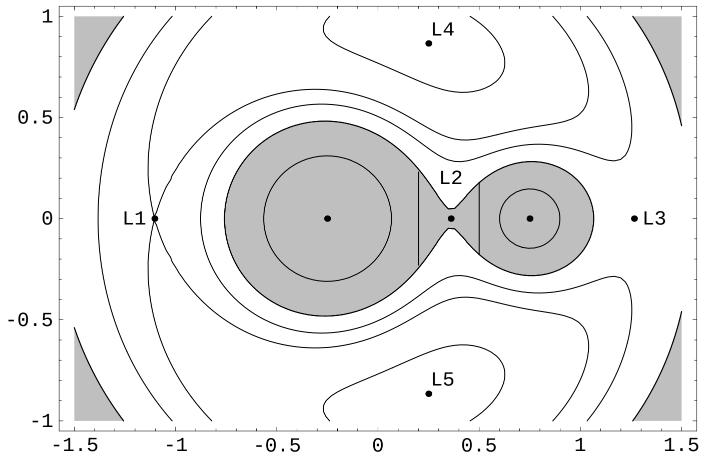

FIGURE 1. Hill's regions and Lagrange points for the planar restricted three-body problem with $\mu = \frac{1}{4}$. Vertical segments $x = a, b$ bound a possible isolating block.

$(-\mu, 0)$ and $(1-\mu, 0)$ in the rotating frame. Then the three-dimensional energy manifolds $\mathcal{M}(\lambda)$ are given by

$$H(x, y, u, v) = \tfrac{1}{2}(u^2 + v^2) - V(x, y) = -\lambda$$

where

$$V(x, y) = \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2}} + \tfrac{1}{2}(x^2 + y^2)$$

is the effective potential function taking into account both gravitational and centrifugal forces. The image of $\mathcal{M}(\lambda)$ under the projection onto configuration space, $\pi(x, y, u, v) = (x, y)$, is called the Hill's region $\mathcal{H}(\lambda)$. Figure 1 shows some typical Hill's regions as well as the five Lagrange points. The shaded one represents a value of λ just below the energy $\bar{\lambda}$ of the collinear Lagrange point, L_2, between the primaries.

Now it is easy to describe Conley's isolating block. Consider the part of the Hill's region which satisfies $a \le x \le b$, where (a, b) contains the x-coordinate of the Lagrange point, $\bar{x}$. For values of λ just below $\bar{\lambda}$ this determines a "tunnel" connecting the two lobes of the Hill's region containing the primaries. The preimage of this tunnel in the energy manifold is a three-dimensional manifold with boundary. This is Conley's isolating block. The boundary surfaces lie over the vertical line segments where $x = a, b$. The required convexity condition is that any solution meeting the boundary tangentially must lie outside the block for all nearby times. This amounts to showing that whenever a solution satifies $x(t) = a, \dot{x}(t) = 0$ then $\ddot{x}(t) < 0$ and that if $x(t) = b, \dot{x}(t) = 0$ then $\ddot{x}(t) > 0$.

Conley verified the convexity condition only for small blocks near the relative equilbrium. One might describe these as infinitesimal isolating blocks. In this case the invariant set in the block can be shown by other methods to be a single periodic orbit. In the three-dimensional restricted problem, the invariant set is homeomorphic to a three-dimensional sphere. In both problems, the invariant set is unstable and has stable and unstable manifolds which are of codimension one in the energy levels. In other words, they form separatrices for the flow. More recently, these separatrices have found application mission design for spacecrafts passing near the Lagrange points [1, 4, 6].

The method for proving existence of the invariant set and its stable and unstable sets is not limited to infinitesimal blocks. If one can verify the required convexity condition for larger blocks with energies far from that of the Lagrange orbit, the conclusion about existence of an invariant set with separatrices still follows. However, it is not clear that the invariant set inside the block retains its former identity as a periodic orbit or invariant three-sphere.

A similar construction can be carried out in the unrestricted planar and spatial three-body problems [5]. Remarkably, the convexity question can still be reduced to the verification of an inequality on a one-dimensional set. This time, it is an arc of great circle on the "shape sphere" of the three-body problem. A brief description of the planar case follows but he reader is referred to [5] for the details of the construction of the blocks. The configuration space of the unrestricted planar three-body problem is six-dimensional since each body has a position vector in $\mathbf{R}^2$. Fixing the center of mass at the origin gives a four-dimensional plane P in $\mathbf{R}^6$. By normalizing the size of the configuration one obtains a normalized configuration space which is a three-dimensional sphere within the plane P. Finally eliminating the rotational symmetry determines a quotient space homeomorphic to a two-dimensional sphere, S, called the *shape sphere*. The last step of eliminating the rotational symmetry is the analogue of working in a rotating coordinate system in the restricted problem. The quotient space is a two-sphere for the same reason this is true for the standard Hopf map of topology. The three-sphere is foliated into circles representing normalized configurations which are rotationally equivalent. The quotient map collapses these circles to points yielding a two-sphere.

The phase space of the unrestricted, planar problem involves the six velocity variables as well. If the total momentum vector is zero, there is a four-dimensional plane of velocities. Fixing the angular momentum, ω, and energy, $h < 0$, determines an integral manifold $\mathcal{M}(h, \omega)$. It turns out that up to equivalence under scaling, the dynamics actually depend on only one parameter which can be taken as $\lambda = \sqrt{2|h|\omega^2}$. After eliminating the rotational symmetry as above, $\mathcal{M}(h, \omega)$ has dimension five. Projecting out the velocities and applying the Hopf map, one obtains a map of $\mathcal{M}(h, \omega)$ onto the two-dimensional shape sphere.

Using this projection of the integral manifolds of the three-body problem onto a sphere, one can find many analogies with the restricted problem. First, one can introduce spherical coordinates (θ, ϕ) on the shape sphere such that the equator, $\phi = 0$, represents the collinear configurations of the three masses. The collinear configurations with m_3 lying between m_1 and m_2 form an arc on the equator of the form $\theta_{13} < \theta < \theta_{23}$ where the endpoints represent collision configurations. The Newtonian potential induces a function $V(\theta, \phi)$ on the shape sphere. There is a unique critical point $\theta_{13} < \bar{\theta} < \theta_{23}$ and this represents a collinear relative equilibrium configuration, L_2, of the three masses. This determines a periodic orbit of the three-body problem where the bodies rotate rigidly around their common center of mass always maintaining this same collinear configuration. There are also analogues of Hill's regions on the shape sphere, namely $\mathcal{H}(\lambda) = \{(\theta, \phi) : V \geq \lambda\}$, and these are the projections of the integral manifolds. Figure 2 shows a

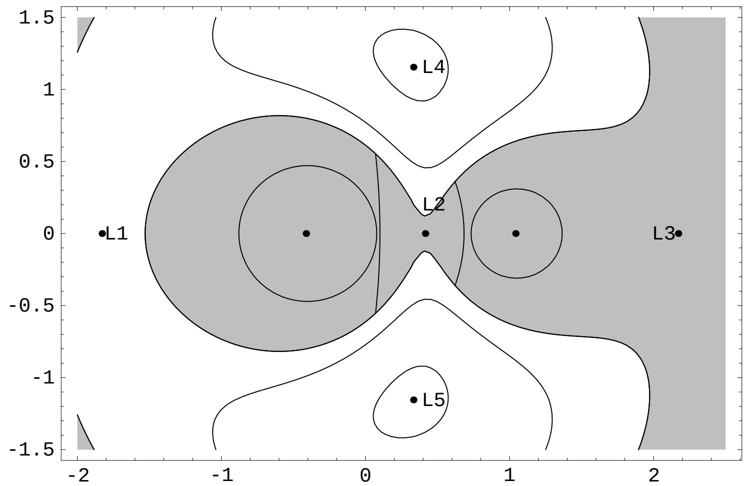

FIGURE 2. Stereographic projection of Hill's regions and Lagrange points on the shape sphere for the unrestricted three-body problem with $m_1 = 3, m_2 = 1, m_3 = 2$. Curves $\theta = a, b$ defining the projection of a possible isolating block are also shown.

stereographic projection of the shape sphere with the equator represented by the horizontal axis.

In light of the obvious similarities between the two problems it is natural to try to construct isolating blocks of the form $a \leq \theta \leq b$. More precisely, the isolating blocks are the five-dimensional manifolds with boundary $B_{(h,\omega)}(a,b) \subset \mathcal{M}(h,\omega)$ which are the preimages of the region in the shape sphere with $a \leq \theta \leq b$. The main result of [5] is to reduce the verification of the convexity of the flow to these blocks to a question about the sign of a certain function along the arcs $\theta = a, b$.

Theorem 1. *$B_{(h,\omega)}(a,b)$ is an isolating block for the flow on $\mathcal{M}(h,\omega)$ provided*

(1) $$V_\theta^2 > V^2 - \lambda^2$$

for all $(\theta, \phi) \in \mathcal{H}(\lambda)$ with $\theta = a, b$ (here $\lambda = \sqrt{2|h|\omega^2}$).

Section 3 is devoted to showing how computational algebra, specifically resultants and Sturm's algorithm, can be used to verify the convexity condition (1).

As in the restricted problem one can also consider the easier question of constructing infinitesimal blocks. For this one only needs to verify an inequality at the relative equilibrium point itself.

Theorem 2. *Consider subsets $B_\epsilon(\xi) \subset \mathcal{M}(h,\omega)$ defined by $\bar{\theta} - \epsilon\xi \leq \bar{\theta} \leq \bar{\theta} + \epsilon\xi$ where $\lambda = \sqrt{2|h|\omega^2} = \bar{\lambda} - \epsilon^2$. Then there exist $\xi > 0, \epsilon_0 > 0$ such that $B_\epsilon(\xi)$ is an isolating block for $0 < \epsilon < \epsilon_0$ provided the following condition holds at the relative equilibrium:*

(2) $$r^2\tau > \tfrac{5}{2}\bar{\lambda}$$

where

$$
\begin{aligned}
r^2 &= \frac{1}{m}(m_1m_2r_{12}^2 + m_1m_3r_{13}^2 + m_2m_3r_{23}^2) \\
\tau &= \frac{m_1+m_2}{r_{12}^3} + \frac{m_1+m_3}{r_{13}^3} + \frac{m_2+m_3}{r_{23}^3} \\
\bar{\lambda} &= \frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}.
\end{aligned} \tag{3}
$$

Whether or not (2) hold depends only on the choice of the masses m_i. It turns out that it holds for some choices of masses and not for others. The boundary between the two kinds of masses will by analyzed using Sturm's algorithm in the next section.

2. The Infinitesimal Block Inequality

In this section, the infinitisimal isolating block inequality (2) will be analyzed. The boundary of the set of masses where the inequality holds is given by setting $2r^2\tau - 5\bar{\lambda} = 0$, where $r^2, \tau, \bar{\lambda}$ are as in (3). These expressions are to be evaluated at the collinear relative equilibrium, L_2.

The location of the relative equilibrium is determined by the following fifth degree equation due to Euler, where ρ denotes the distance ratio r_{13}/r_{12}:

$$
\begin{aligned}
&(m_1+m_2)\rho^5 - (2m_1+3m_2)\rho^4 + (m_1+3m_2+2m_3)\rho^3 \\
&\qquad - (m_1+3m_3)\rho^2 + (2m_1+3m_3)\rho - (m_1+m_3) = 0.
\end{aligned} \tag{4}
$$

The other distance ratio is $r_{23}/r_{12} = 1-\rho$.

Using the normalization $r_{12} = 1$ one finds

$$
\begin{aligned}
r^2 &= \frac{1}{m}(m_1m_2 + m_1m_3\rho^2 + m_2m_3(1-\rho)^2) \\
\tau &= (m_1+m_2) + \frac{m_1+m_3}{\rho^3} + \frac{m_2+m_3}{(1-\rho)^3} \\
\bar{\lambda} &= m_1m_2 + \frac{m_1m_3}{\rho} + \frac{m_2m_3}{(1-\rho)}.
\end{aligned}
$$

Setting $2r^2\tau - 5\bar{\lambda} = 0$ and clearing denominators gives another polynomial equation for ρ. Taking the resultant of this equation and Euler's equation and dropping some positive factors gives the homogeneous polynomial equation $R = 0$ where R is given by the following formula:

$$
\begin{aligned}
&125\,m_1^6m_2^2 - 284\,m_1^5m_2^3 + 526\,m_1^4m_2^4 - 284\,m_1^3m_2^5 + 125\,m_1^2m_2^6 \\
&+ 250\,m_1^6m_2m_3 - 978\,m_1^5m_2^2m_3 + 72\,m_1^4m_2^3m_3 + 72\,m_1^3m_2^4m_3 \\
&- 978\,m_1^2m_2^5m_3 + 250\,m_1m_2^6m_3 + 125\,m_1^6m_3^2 - 222\,m_1^5m_2m_3^2 \\
&- 1775\,m_1^4m_2^2m_3^2 + 7560\,m_1^3m_2^3m_3^2 - 1775\,m_1^2m_2^4m_3^2 - 222\,m_1m_2^5m_3^2 \\
&+ 125\,m_2^6m_3^2 - 116\,m_1^5m_3^3 + 216\,m_1^4m_2m_3^3 - 2760\,m_1^3m_2^2m_3^3 \\
&- 2760\,m_1^2m_2^3m_3^3 + 216\,m_1m_2^4m_3^3 - 116\,m_2^5m_3^3 + 718\,m_1^4m_3^4 \\
&+ 1720\,m_1^3m_2m_3^4 - 1887\,m_1^2m_2^2m_3^4 + 1720\,m_1m_2^3m_3^4 + 718\,m_2^4m_3^4 \\
&- 436\,m_1^3m_3^5 + 958\,m_1^2m_2m_3^5 + 958\,m_1m_2^2m_3^5 - 436\,m_2^3m_3^5 - 83\,m_1^2m_3^6 \\
&- 106\,m_1m_2m_3^6 - 83\,m_2^2m_3^6 - 32\,m_1m_3^7 - 32\,m_2m_3^7 = 0
\end{aligned}
$$

This is an eighth-degree algebraic curve in the projective plane determined by the masses. However its intersection with the set of real positive masses turns out to be fairly simple. Moreover, the inequality $R > 0$ is equivalent to (2) and provides an immediate test for whether a given set of masses admits infinitisimal blocks.

To investigate the curve $R = 0$, it is convenient to eliminate one variable by normalizing the masses. Assume that $m_{12} = m_1 + m_2 = 1$ and introduce two parameters $u, v \in \mathbf{R}$ such that

$$m_1 = \frac{1}{2} - u \quad m_2 = \frac{1}{2} + u \quad m_3 = v, \qquad u = \frac{m_2 - m_1}{2m_{12}} \quad v = \frac{m_3}{m_{12}}.$$

Then the parameter values corresponding to nonnegative masses are $-\frac{1}{2} \leq u \leq \frac{1}{2}$ and $0 \leq v < \infty$. It is also convenient to replace the distance ρ by a parameter $z = \frac{1}{2} - \rho$ so that the mutual distances become

$$r_{12} = 1 \qquad r_{13} = \frac{1}{2} - z \qquad r_{23} = \frac{1}{2} + z.$$

The distance ρ satisfies $0 < \rho < 1$ and so z satisfies $-\frac{1}{2} < z < \frac{1}{2}$.

In the new variables, Euler's equation becomes

$$7\,u - 17\,z - 24\,v\,z + 40\,u\,z^2 + 8\,z^3 - 32\,v\,z^3 - 16\,u\,z^4 - 16\,z^5 = 0$$

and the equation $2r^2\tau - 5\bar{\lambda} = 0$ becomes

$$\begin{aligned}
&13 - 52\,u^2 + 5\,v - 108\,u^2\,v - 8\,v^2 - 192\,u\,z + 768\,u^3\,z + 112\,u\,v\,z + \\
&416\,u\,v^2\,z + 228\,z^2 - 912\,u^2\,z^2 + 1004\,v\,z^2 - 3312\,u^2\,v\,z^2 + 832\,v^2\,z^2 - \\
&256\,u\,z^3 + 1024\,u^3\,z^3 - 960\,u\,v\,z^3 + 1792\,u\,v^2\,z^3 - 144\,z^4 + 576\,u^2\,z^4 - \\
&112\,v\,z^4 - 1088\,u^2\,v\,z^4 + 896\,v^2\,z^4 + 2304\,u\,v\,z^5 + 2560\,u\,v^2\,z^5 + 192\,z^6 - \\
&768\,u^2\,z^6 + 576\,v\,z^6 - 1280\,u^2\,v\,z^6 - 1024\,u\,v\,z^7 - 512\,v\,z^8 = 0
\end{aligned}$$

Together, these equations define a curve C in (u, v, z) space. The projection of this curve onto various planes can be found using resultants. If the curve is projected onto the (u, z) plane the equation becomes:

$$\begin{aligned}
&(u + z)\,(-1 + 2\,z)^2\,(1 + 2\,z)^2\,\left(7 - 40\,u\,z + 32\,u\,z^3 + 16\,z^4\right) \\
&\left(-7\,u + 56\,z - 308\,u\,z^2 + 136\,z^3 - 272\,u\,z^4 + 64\,z^5 + 320\,u\,z^6 + 128\,z^7\right) = 0.
\end{aligned}$$

The factorization shows that C is reducible, i.e., it is a union of several separate irreducible plane curves called the components of C , each given by one of the factors. The first three factors define lines which lie outside the physical range of parameters. Remarkably, the other two factors are linear in u and so on each of the corresponding components, u can be written as a rational function of z. The fourth factor leads to a rational expression for u which takes values outside the physical range of parameters. This leaves only the last factor which leads to the formula:

$$u = \frac{8\,z\,\left(7 + 17\,z^2 + 8\,z^4 + 16\,z^6\right)}{7 + 308\,z^2 + 272\,z^4 - 320\,z^6} \tag{5}$$

Substituting this expression into Euler's equation leads to an equation which can be solved for v with the result:

$$v = \frac{(-1 + 2\,z)^2\,(1 + 2\,z)^2\,\left(91 - 56\,z^2 + 48\,z^4\right)}{8\,\left(7 + 308\,z^2 + 272\,z^4 - 320\,z^6\right)} \tag{6}$$

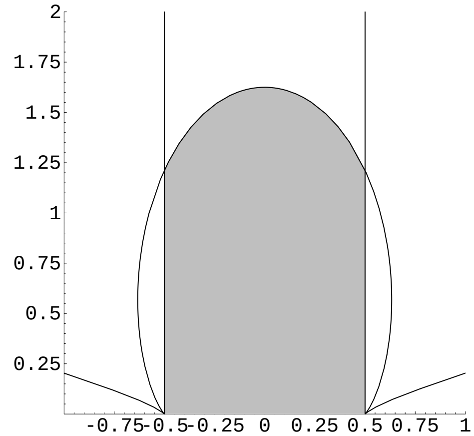

FIGURE 3. Masses for which infinitisimal isolating blocks B_ϵ exist lie below the Ω curve in the vertical strip. The variables are $u = \frac{m_2 - m_1}{2(m_1+m_2)}$ and $v = \frac{m_3}{m_1+m_2}$.

The resultant of these two equations with respect to z gives an implicit formula for the projection of this component of the curve into the (u, v)-plane:

$$\begin{aligned}
&13 + 8\,u^2 + 656\,u^4 - 8960\,u^6 + 21504\,u^8 - 164\,v + \\
&1056\,u^2\,v + 15424\,u^4\ - 68096\,u^6\,v + 954\,v^2 - 17600\,u^2\,v^2 + \\
&128800\,u^4\,v^2 - 166656\,u^6\,v^2 - 2660\,v^3 + 19168\,u^2\,v^3 - \\
&63808\,u^4\,v^3 + 2989\,v^4 + 49560\,u^2\,v^4 - 62256\,u^4\,v^4 + 2088\,v^5 - \\
&36256\,u^2\,v^5 - 1088\,v^6 - 960\,u^2\,v^6 - 512\,v^7 = 0.
\end{aligned}$$

The parametrization given by (5) and (6) shows that this irreducible, degree eight, plane algebraic curve is rational.

Figure 3 shows a portion of the curve shaped like the Greek letter Ω. The rounded central part connecting the cusps at $(\pm\frac{1}{2}, 0)$ corresponds to parameters $-\frac{1}{2} < z < \frac{1}{2}$. The vertical lines at $u = \pm\frac{1}{2}$ mark the boundaries of the strip where the masses are nonnegative. Evidently, the equation $2r^2\tau - 5\bar{\lambda} = 0$ determines a convex curve which cuts the strip into two parts. It is not hard to see that the part of the strip below the curve is the one where the isolating block inequality $r^2\tau > \frac{5}{2}\bar{\lambda}$ holds. In particular, it holds whenever the value of $v = \frac{m_3}{m_1+m_2}$ is less than a certain constant $v_0 \approx 1.21$. For all masses corresponding to this region, proposition 2 shows that the isolating blocks B_ϵ can be constructed for all sufficiently small ϵ.

Next it will be shown how the properties of the omega curve can be rigorously verified using the parametrization and Sturm's algorithm. Let $u(z), v(z)$ denote the rational functions (5), (6) which parametrize the curve. By symmetry, it suffices to consider the right side of the omega curve, which corresponds to the parameter interval $0 \le z \le \frac{1}{2}$.

One finds $(u(0), v(0)) = (0, \frac{13}{8})$ and $(u(\frac{1}{2}), v(\frac{1}{2})) = (\frac{1}{2}, 0)$. Next, it will shown that the part of the curve between these points can be viewed as a graph of a convex function $u = \phi(v), 0 \le v \le \frac{13}{8}$. To see that the curve can be parametrized by v it suffices to show that $v'(z) < 0$ for $0 < z < \frac{1}{2}$. The formula for the derivative $v'(z)$ is a rational function of z with integer coefficients which vanishes at $z = 0, \frac{1}{2}$. After cancelling factors of z and $2z - 1$ one obtains another rational function with integer coefficients. Sturm's algorithm can be applied to the polynomials in the numerator and denominator to show that they have no real roots in the parameter range $0 \le z \le \frac{1}{2}$. Recall that Sturm's algorithm gives an effective count of the number of real roots of a real polynomial in any interval [7]. The algorithm involves computing a sequence of polynomials and checking the difference of the numbers of sign changes of the sequence at the two endpoints of the interval in question. If the polynomials have integer or rational coefficients and if the endpoints of the interval are also rational, these computations can all be carried out exactly by computer. From the nonvanishing of $v'(z)$ it follows that the curve can be parametrized (in the opposite sense) by v. Let $u = \phi(v)$ be be the resulting function. To see that $\phi(v)$ is convex, note that its derivative $\phi'(v)$ is given by the rational function $\sigma(z) = u'(z)/v'(z)$ and $\sigma'(z)$ is yet another rational function. Applying Sturm's algorithm to the numerators and denominators of these functions one can show that the slope $\sigma(z)$ has one root and that $\sigma'(z)$ has no roots for $0 < z < \frac{1}{2}$. It follows that $u = \phi(v)$ is convex with a single maximum for $0 < v < \frac{13}{8}$.

Finally, one can estimate where the curve leaves the strip of nonnegative masses by looking for the solutions of $u(z) = \frac{1}{2}$. Aside from the root at the cusp $(u, v) = (\frac{1}{2}, 0)$ Sturm shows that there is exactly one more root which can be isolated in the parameter interval $\frac{782}{10000} < z < \frac{783}{10000}$ which gives $v_0 \approx 1.21$.

3. The Global Block Inequality

The more complicated inequality (1) can be used to study existence of larger isolating blocks which extend far from the relative equilibrium in both the configuration space and the parameter space. This inequality has to be checked not just at the relative equilibrium point, but along arcs of the shape sphere of the form $\theta = c$. Whether or not the inequality holds will depend on the parameters $c, \lambda, m_1, m_2, m_3$. Unfortunately, it is not possible to give a comprehensive description of the boundary of the subset of the five-dimensional parameter space where the inequality holds. However, it is possible to verify the inequality for fixed values of the parameters.

First imagine fixing the masses. Then the problem is to find (c, λ) such that

$$\lambda^2 > V^2(c, \phi) - V_\theta^2(c, \phi) \tag{7}$$

holds for all ϕ such that (c, ϕ) is in the Hill's region $\mathcal{H}(\lambda)$ (actually the inequality automatically holds outside the Hill's region since, by definition, $V < \lambda$ there). Now the parameter $\lambda = \sqrt{2|h|\omega^2}$ is positive and only values below the level of the relative equilibrium are of interest since for $\lambda > \bar{\lambda}$ there is no "tunnel" to contain an invariant set.

The plot in the (c, λ) plane showing the points with $0 < \lambda < \bar{\lambda}$ where (7) holds will be called an *isolating block bifurcation diagram* for the given masses. The right column of figure 4 shows three numerically computed examples. In both cases, the region of validity of the inequality is bounded below by a continuous graph. In fact this is true quite generally. To see this, consider a rectangle $c_1 \le c \le c_2$ and $\lambda_0 \le \lambda \le \bar{\lambda}$. For each fixed $c \in [c_1, c_2]$ choose an interval $\phi_1(c) \le \phi \le \phi_2(c)$ large enough to contain the part of the arc $\theta = c$ in all the Hill's regions $\mathcal{H}(\lambda)$,

$\lambda_0 \leq \lambda \leq \bar{\lambda}$. Let $W(c) = \max(V^2(c,\phi) - V_\theta^2(c,\phi))$ where the maximum is taken over the chosen interval. Then $W(c) > 0$ is a continuous function and the graph $\lambda = \sqrt{W(c)}$ with $\lambda_0 \leq \lambda \leq \bar{\lambda}$ forms the boundary of the isolating block bifurcation diagram in the rectangle. For each fixed c it is easy to estimate $W(c)$ numerically and that is essentially how the bifurcation diagrams in figure 4 were produced.

In practice, however, it is inconvenient to work with spherical coordinates on the shape sphere. For both numerical and rigorous, algebraic computations, an alternative version of the convexity condition will first be given in Jacobi coordinates, and then expressed in terms of polynomials in the mutual distances r_{ij}. For simplicity, only the planar case will be described here.

If the Cartesian position vectors of the three bodies are $q_i \in \mathbf{R}^2$ then the Jacobi coordinates are $x = q_2 - q_1$ and $y = q_3 - c_{12}$ where c_{12} is the center of mass of the masses m_1 and m_2. The center of mass of the whose system is assumed to be at the origin. The formulas for the Newtonian potential and moment of inertia are:

$$\begin{aligned} r^2 &= \mu_1|x|^2 + \mu_2|y|^2 \\ U(x,y) &= \frac{m_1m_2}{|x|} + \frac{m_1m_3}{|y+\nu_2x|} + \frac{m_2m_3}{|y-\nu_1x|} \end{aligned}$$

where

$$\begin{aligned} m_{12} &= m_1 + m_2 & m &= m_1 + m_2 + m_3 \\ \nu_1 &= \frac{m_1}{m_{12}} & \nu_2 &= \frac{m_2}{m_{12}} \\ \mu_1 &= \frac{m_1m_2}{m_{12}} & \mu_2 &= \frac{m_{12}m_3}{m} \end{aligned} .$$

Since the center of mass has been eliminated, the Jacobi configuration space $(x,y) \in \mathbf{R}^4$ represents the four-dimensional plane, P, of the introduction. Setting $r^2 = 1$ defines the three-sphere of normalized configurations. Finally, the Hopf map from this sphere to the shape sphere is given by $h(x,y) = (h_1(x,y), h_2(x,y), h_3(x,y))$ where

$$\begin{aligned} h_1(x,y) &= \mu_1|x|^2 - \mu_2|y|^2 \\ h_2(x,y) &= 2\sqrt{\mu_1\mu_2}\, x\cdot y \\ h_3(x,y) &= 2\sqrt{\mu_1\mu_2}\,(x_1y_2 - x_2y_1). \end{aligned} \tag{8}$$

It can be shown that $|h(x,y)|^2 = r^4$ so h maps the normalized configuration three-sphere onto the unit two-sphere in $\mathbf{R}^3$. Note that fixing a value $\theta = c$ on the shape sphere is equivalent to setting

$$h_2(x,y) - \tan(c)h_1(x,y) = 0.$$

Finally, the convexity condition in Jacobi coordinates turns out to be:

$$\lambda^2 > U^2 - \tfrac{1}{4}\tilde{D}U^2 \tag{9}$$

where

$$\tilde{D}U = h_1\left(\sqrt{\tfrac{\mu_1}{\mu_2}}\, x\cdot U_y + \sqrt{\tfrac{\mu_2}{\mu_1}}\, y\cdot U_x\right) - h_2\left(x\cdot U_x - y\cdot U_y\right).$$

The mutual distances r_{12}, r_{13}, r_{23} are rotation invariant and so provide convenient coordinates on the shape sphere. The moment of inertia is given by

$$r^2 = \frac{1}{m}(m_1m_2r_{12}^2 + m_1m_3r_{13}^2 + m_2m_3r_{23}^2)$$

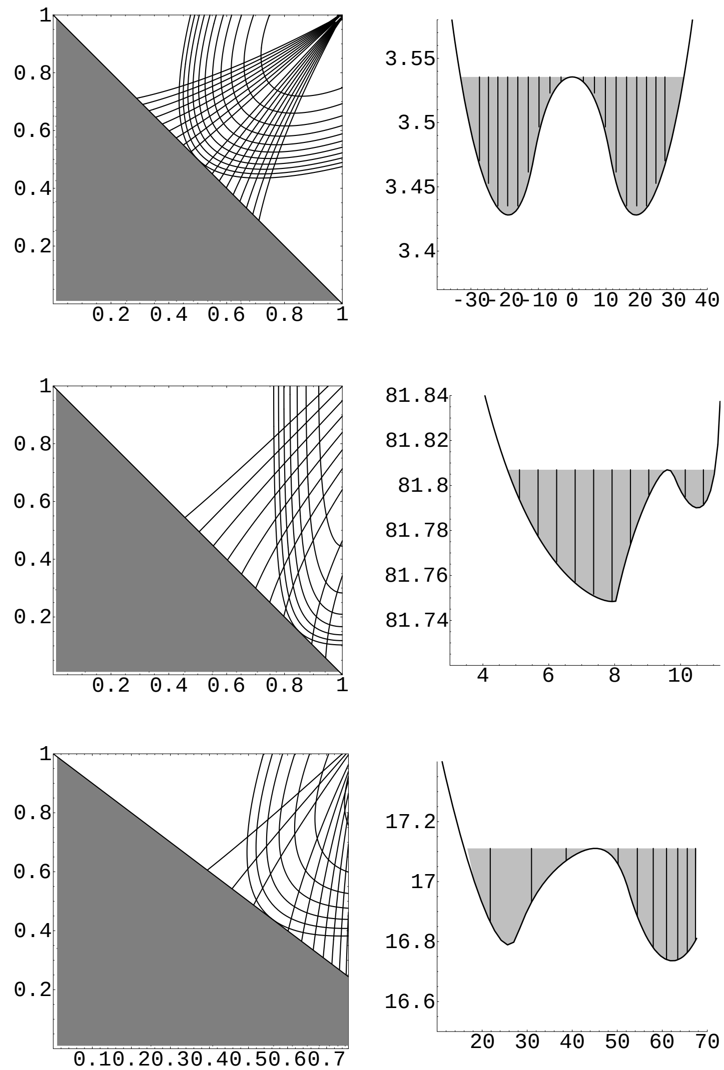

FIGURE 4. Computation of isolating block bifurcation diagrams. The left column shows triangles, T, in the (r_{13}, r_{23})-plane where the isolating block inequality is analyzed. The curves sloping from southwest to northeast are branches of hyperbolas representing several choices of $\theta = c$. Some level curves of the potential are also shown. The right column shows numerically computed isolating block bifurcation diagrams in the (c, λ)-plane with c in degrees. The vertical bars represent values which have been rigorously verified using Sturm's algorithm. From top to bottom the masses are $(m_1, m_2, m_3) = (1, 1, 1)$, $(81, 1, \frac{1}{100})$, and $(3, 1, 2)$.

and the equation determining the arc $\theta = c$ becomes:

$$q = h_2 - \tan(c) h_1 = 0 \tag{10}$$

where

$$h_1(x,y) = 2\mu_1 r_{12}^2 - r^2$$
$$h_2(x,y) = \sqrt{\mu_1\mu_2}\left((\nu_1 - \nu_2) r_{12}^2 + r_{13}^2 + r_{23}^2\right).$$

The Newtonian potential U is clearly a rational function of the r_{ij} and with some effort one can show that

$$\tilde{D}U = \frac{\sqrt{\mu_1\mu_2}\, m r^2 h_1}{m_{12}}\left(\frac{1}{r_{23}^3} - \frac{1}{r_{13}^3}\right) + \frac{r^2 h_2}{2}\left(\frac{m_{12}}{r_{12}^3} + \frac{m_2 m_3 - m_1 m}{m_{12}\, r_{13}^3} + \frac{m_1 m_3 - m_2 m}{m_{12}\, r_{23}^3}\right).$$

Thus, after clearing denominators, all of the relevant equations can be expressed in terms of polynomials in the mutual distances.

Instead of working on the sphere $r^2 = 1$ one can multiply U and $\tilde{D}U$ by appropriate powers of r to obtain homogeneous rational functions of degree 0. Then, as in the last section, the simpler normalization $r_{12} = 1$ can be used. With this approach, there are only two configuration variables, r_{13} and r_{23}. For the purposes of studying the neighborhood of the collinear Lagrange point, it is convenient to restrict attention to a certain triangle T in the (r_{13}, r_{23}) plane (see the left column of figure 4). Clearly $r_{ij} \geq 0$. The triangle inequality implies $1 \leq r_{13} + r_{23}$ and the diagonal line where equality occurs corresponds to the collinear configurations with mass m_3 between m_1 and m_2. Imposing upper bounds $r_{13} \leq \bar{r}_{13}$ and $r_{23} \leq \bar{r}_{23}$ determines a triangle, T. The upper bounds can be chosen so that the level curves of constant θ and U meet transversely in T, as shown in the figure.

To see this, note that for fixed c the curve (10) representing $\theta = c$ can be put in the form given by

$$A r_{13}^2 - B r_{23}^2 = C$$

where

$$A = m_{12}(\sqrt{\mu_1\mu_2}\, m + \tan(c) m_1 m_3)$$
$$B = m_{12}(\sqrt{\mu_1\mu_2}\, m - \tan(c) m_2 m_3)$$
$$C = m(\tan(c) m_1 m_2 - \sqrt{\mu_1\mu_2}(m_1 - m_2)).$$

It is convenient to restrict the values of c so that $A \geq 0$ and $B \geq 0$. Let c_1 and c_2 be the angles making $A = 0$ and $B = 0$ respectively. Then the curves $\theta = c$, $c_1 < c < c_2$, are branches of hyperbolas. The limiting equations for $c = c_1$ and $c = c_2$ define lines $r_{23} = \sqrt{-C_1/B_1}$ and $r_{13} = \sqrt{C_2/A_2}$ respectively, where A_i, B_i, C_i denote the values of A, B, C when $c = c_i$. The upper bounds defining T will be chosen as $\bar{r}_{13} = \min(\sqrt{C_1/A_1}, 1)$ and $\bar{r}_{23} = \min(\sqrt{-C_1/B_1}, 1)$. In T, the hyperbolas $\theta = c$ cross the level curves of the U monotonically. To see this it suffices to show that the gradients of U and q are never parallel, i.e., that the Jacobian determinant of (U, q) with respect to (r_{13}, r_{23}) never vanishes in T. This determinant turns out to be

$$\frac{2 A m_2 m_3}{r_{23}^3} + \frac{2 B m_1 m_3}{r_{13}^3}$$

which is clearly positive in T.

Now fix $c \in (c_1, c_2)$ and consider the part of the hyperbola $h_2 - \tan(c)h_1 = 0$ which lies in T. Suppose this hyperbola crosses all of the level curves $U = \lambda$ for $\lambda_0 \leq \lambda \leq \bar{\lambda}$. Then for these values of λ, the part of the hyperbola in T contains representatives of all the points on the arc $\theta = c$ which lie in the Hill's region $\mathcal{H}(\lambda)$. Let $W(c)$ be the maximum of the function $U^2 - \frac{1}{4}\tilde{D}U^2$ along the part of the hyperbola in T. Then $\lambda = \sqrt{W(c)}$ gives a boundary point of the isolating block bifurcation diagram provided $\lambda_0 \leq \sqrt{W(c)} \leq \bar{\lambda}$. One can estimate $W(c)$ numerically just by evaluation of rational functions at many points on the hyperbola. This is how the diagrams in figure 4 were produced.

Finally, one can combine these ideas with Sturm's algorithm to provide rigorous verification of the convexity condition. This time imagine that both c and λ are fixed. Assume that $\theta = c$ determines a piece of hyperbola in T which crosses the level curves $U = \lambda$ for $\lambda_0 \leq \lambda \leq \bar{\lambda}$ and that the chosen λ is in $[\lambda_0, \bar{\lambda}]$. Then one endpoint of the hyperbola lies outside the Hill's region $\mathcal{H}(\lambda)$ so the inequality (9) holds there. If it can be shown that the function $P = \lambda^2 - U^2 + \frac{1}{4}\tilde{D}U^2$ has no zeros on the hyperbolic segment, then the inequality must hold on the whole segment. Thus the convexity condition will be verified for this choice of (c, λ) and hence also for the whole line segment (c, λ'), $\lambda \leq \lambda' \leq \bar{\lambda}$.

After clearing denominators, P becomes a polynomial in r_{13}, r_{23} and the hyperbola corresponding to $\theta = c$ is given by a second polynomial in the distances. Taking the resultant of these two polynomials with respect to one of the variables gives a polynomial, R, in the remaining variable. Suppose, for example that r_{23} is eliminated and that $R(r_{13})$ is the resultant. The part of the hyperbola in T determines an interval $a \leq r_{13} \leq b$. If $R(r_{13})$ has no real roots in this interval, then the convexity condition will be verified. This can be rigorously checked with Sturm's algorithm provided the masses m_i and the values of $\tan(c)$ and λ are rational numbers, or at least algebraic numbers which are exactly representable in the computer.

Unfortunately, the polynomials involved are so complicated that these computations can be carried out only after fixing the masses and parameters. Even then the polynomials are formidable. For example, in the simple case $(m_1, m_2, m_3) = (3, 1, 2)$, $\lambda = \frac{452}{27}$, $\tan(c) = \frac{9}{5}$ the resultant R is:

$$
\begin{aligned}
&729028394826491930625 - 224188248360709342710000\,x^2 \\
&- 20584331148042125100000\,x^3 + 305164415211685934439600\,x^4 \\
&+ 614403564404713024032000\,x^5 - 245564383188068045084880\,x^6 \\
&- 8130243120997900764326400\,x^7 + 131382563572585023720652800\,x^8 + \\
&\vdots \\
&- 2766282263977786002175426560\,x^{29} - 2037680048246567368812134400\,x^{30} \\
&+ 1157591356194848328017510400\,x^{31} + 5383562340067454052886118400\,x^{32} \\
&- 2191257001307356256010240000\,x^{33} - 2179287950459879099990016000\,x^{34} \\
&+ 3866924119954158098841600\,x^{36}
\end{aligned}
$$

where $x = r_{13}$ and twenty more terms have been omitted.

The chosen value of $\tan(c)$ represent $\theta = c \approx 61^\circ$. This choice determines one of the hyperbolic segments in the triangle T in the lower left of figure 4. This segment lies over the r_{13}-interval $[\frac{\sqrt{106}-2}{12}, \frac{2}{\sqrt{7}}]$. Sturm's algorithm shows that R has no real roots for $x = r_{13}$ in this interval. Therefore the isolating block inequality is verified on the arc $\theta = c$ for the given values of m_i and λ. The segment $\{(c, \lambda') : \lambda \leq \lambda' \leq \bar{\lambda}\}$ then follows as described above. This verification is indicated by one of the vertical line segments in the lower right isolating block bifurcation diagram of figure 4. Similarly, the other segments superimposed on the numerically computed isolating block bifurcation diagrams were verified by applying Sturm's algorithm to the point (c, λ) at the bottom of the segment. Although this approach does provide a rigorous method for verifying that the isolating block inequalities hold in particular cases, it does not allow one to understand the whole boundary curve $\lambda = \sqrt{W(c)}$ or to study how the curve changes as the masses change.

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