A TOPOLOGICAL EXISTENCE PROOF FOR THE SCHUBART ORBITS IN THE COLLINEAR THREE-BODY PROBLEM

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To Carles Simó on his 60th birthday – molts anys

Abstract. A topological existence proof is presented for certain symmetrical periodic orbits of the collinear three-body problem with two equal masses, called Schubart orbits. The proof is based on the construction of a Wazewski set $\mathcal{W}$ in the phase space. The periodic orbits are found by a shooting argument in which symmetrical initial conditions entering $\mathcal{W}$ are followed under the flow until they exit $\mathcal{W}$. Topological considerations show that the image of the symmetrical entrance states under this flow map must intersect an appropriate set of symmetrical exit states.

1. Introduction

The collinear three-body problem describes the motion of three point masses moving on a line under the influence of their mutual gravitational attraction. It was first studied by Euler [6] who was able to find simple explicit solutions – the first solutions of the three-body problem ever found. Euler's solutions are homothetic; the configuration formed by the three masses shrinks while maintaining the same shape. The solutions end in triple collision.

It is intuitively clear that double and triple collisions will play an important role in the problem. It is possible to regularize the double collisions so that the colliding particles bounce and then move apart. However, McGehee [8] showed that there is no such regularization for the triple collision. Instead, he devised a system of coordinates and a change of time scale under which motions which previously ended in triple collision at some finite time, now approach an equilibrium point as the rescaled time tends to infinity. By combining the regularization of double collisions with the blow-up of triple collisions one obtains a dynamical system without singularities. It is this modified collinear three-body problem which will be studied here.

The goal of this paper is to give a topological proof of existence of certain simple, symmetric periodic motions in the case where two of the three masses are equal. The behavior of these orbits is shown in figure 1. Initially, the unequal mass, m_3, is at the midpoint of the equal ones. This is Euler's central configuration. If the particles were released with zero velocity, a homothetic collapse to triple collision

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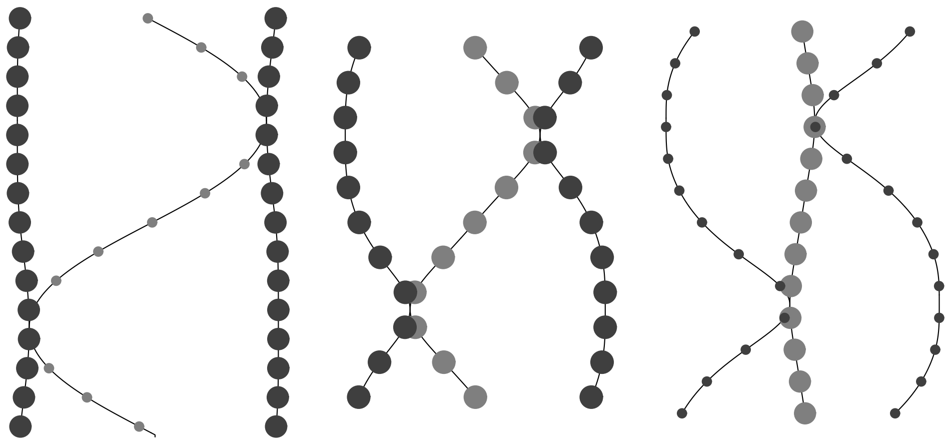

FIGURE 1. Symmetrical periodic solution for $m_3 = \frac{1}{10}, 1, 10$. The orbit are collinear but center of mass has been given a small downward drift.

would ensue. However, the initial velocities are chosen so that m_2 and m_3 approach one another while m_1 moves in the opposite direction. The main point of the paper is to show that it is possible to choose the velocities so that m_1 slows to a stop at exactly the same moment when m_2 and m_3 collide at $u = \frac{\pi}{2}$. This represents the first quarter period of the periodic orbit. During the next quarter period, the particles return to the initial configuration but with the velocities reversed. Finally, this sequence is repeated on the other side with m_1 and m_3 colliding. Such orbits were found numerically by Schubart [10] (see also [7]). The author is not aware any previously published existence proof, however.

Perhaps of greater interest than the orbits themselves is the method of proof, which is a variation of an idea used by Conley in the restricted three-body problem [1]. In Conley's paper, the retrograde lunar orbit of Hill is shown to exist for a wide range of values of the Jacobi constant. After regularizing double collisions, the problem becomes one of finding a solution of a system of second-order differential equations in the plane which moves from the positive x-axis to the positive y-axis across the first quadrant and meets both axes orthogonally.

Such a solution is found by a kind of shooting argument. Points starting orthogonal to the x-axis are followed until one of two exit conditions holds – they either hit the positive y-axis or their velocity vectors become horizontal. As the initial point along the x-axis varies, the final behavior changes from hitting the y-axis with nonzero slope to having a horizontal velocity vector before hitting the positive y-axis. Somewhere in between, there must be a point whose velocity becomes horizontal exactly when it reaches the positive y-axis and this gives the desired periodic solution. The main difficulty in justifying such an argument is to show that the initial solutions really arrive at one of the two kinds of final states, and that the final state depends continously on the initial condition. For this, Conley constructed an isolating block.

Isolating blocks were developed by Conley and Easton as a way to defining a topological index for invariant sets [2, 3, 5]. Among their useful properties is the fact for initial conditions which leave the isolating block, the amount of time required to leave depends continuously on initial conditions. It follows that the location of the exit point also varies continuously. In [1] Conley constructs an isolating block in a manifold of fixed Jacobi constant and uses it to justify the shooting argument outlined above.

The concept of isolating block is related to earlier ideas of Wazewski [11]. It is possible to get the crucial property of continous exit times under weaker assumptions than are needed for the topological index theory. For example, whereas isolating blocks are always compact, Wazewski sets need not be. In this paper the proof will be based on the construction of a Wazewski set, rather than an isolating block. It is hoped that Wazewski sets and higher-dimensional shooting arguments will provide existence proofs for symmetrical periodic solutions in more complicated systems, such as the planar three-body problem or symmetrical subsystems of the n-body problem.

2. Equations of Motion and Regularization

Consider the collinear three-body problem with two equal masses $m_1 = m_2 = 1$ and an arbitrary mass $m_3 > 0$. Let the positions be $q_i \in \mathbf{R}$ and the velocities be $v_i = \dot{q}_i \in \mathbf{R}$. The Newton's laws of motion are the Euler-Lagrange equation of the Lagrangian

$$L = \tfrac{1}{2}K + U \tag{1}$$

where

$$\begin{aligned} K &= v_1^2 + v_2^2 + m_3 v_3^2 \\ U &= \frac{1}{r_{12}} + \frac{m_3}{r_{13}} + \frac{m_3}{r_{23}}. \end{aligned} \tag{2}$$

Here $r_{ij} = |q_i - q_j|$ denotes the distance between the i-th and j-th masses. The total energy of the system is constant:

$$\tfrac{1}{2}K - U = h.$$

Assume without loss of generality that total momentum is zero and that the center of mass is at the origin, i.e.,

$$v_1 + v_2 + m_3 v_3 = q_1 + q_2 + m_3 q_3 = 0.$$

Introduce Jacobi variables

$$\xi_1 = q_2 - q_1 \qquad \xi_2 = q_3 - \tfrac{1}{2}(q_1 + q_2)$$

and their velocities $\eta_i = \dot{\xi}_i$. Then the equations of motion are given by a Lagrangian of the same form (1) where now

$$\begin{aligned} K &= \tfrac{1}{2}|\eta_1|^2 + \mu|\eta_2|^2 \\ U &= \frac{1}{r_{12}} + \frac{m_3}{r_{13}} + \frac{m_3}{r_{23}} \end{aligned} \tag{3}$$

and $\mu = \frac{2m_3}{2+m_3}$. The mutual distances are given by

$$\begin{aligned} r_{12} &= |\xi_1| \\ r_{13} &= |\xi_2 + \tfrac{1}{2}\xi_1| \\ r_{23} &= |\xi_2 - \tfrac{1}{2}\xi_1|. \end{aligned} \tag{4}$$

The use of Jacobi coordinates eliminates the translational symmetry of the problem and reduces the number of degrees of freedom from 3 to 2.

The next step is to replace (ξ_1, ξ_2) by variables representing the size and shape of the configuration. The approach used here is the collinear version of the Hopf-map reduction of the planar three-body problem [9]. Define r, w_1, w_2 via:

$$\begin{aligned} r^2 &= \tfrac{1}{2}\xi_1^2 + \mu\xi_2^2 \\ w_1 &= \tfrac{1}{4}\xi_1^2 - \tfrac{1}{2}\mu\xi_2^2 \\ w_2 &= \sqrt{\tfrac{\mu}{2}}\,\xi_1\xi_2 \end{aligned} \tag{5}$$

It is easy to check that the new variables satisfy the relation:

$$w_1^2 + w_2^2 = \tfrac{1}{4}I^2 = \tfrac{1}{4}r^4.$$

The quantity $I = r^2$ is the moment of inertia, a convenient measure of the overall size of the configuration formed by the three bodies. Its shape can be represented by the normalized vector $s = 2w/r^2$ which lies in the unit circle $\mathbf{S}^1$. It will be convenient to replace s by an angle θ such that $s = (\cos\theta, \sin\theta)$. The angle describes the shape of the collinear configuration. The shapes which are relevant here are those for which the mass m_3 lies between the equal masses m_1, m_2. This corresponds to the interval $-\theta^* \le \theta \le \theta^*$ where

$$\theta^* = \tan^{-1}\sqrt{m_3(2+m_3)}.$$

It is easy to see that θ^* increases from 0 to $\frac{\pi}{2}$ as m_3 increases from 0 to ∞.

The variables r, θ satisfy the Euler-Lagrange equations of the Lagrangian

$$L = \tfrac{1}{2}K + \tfrac{1}{r}W(\theta) \tag{6}$$

where:

$$\begin{aligned} K &= \dot{r}^2 + \tfrac{1}{4}r^2\dot{\theta}^2 \\ W &= \frac{1}{\rho_{12}} + \frac{m_3}{\rho_{13}} + \frac{m_3}{\rho_{23}}. \end{aligned} \tag{7}$$

Here $\rho_{ij} = r_{ij}/r$ is the normalized interparticle distance, i.e., the interparticle distance after scaling the configuration to have moment of inertia 1. After some computation using (4) and (5) and some trigonometric identities, one finds

$$\begin{aligned} \rho_{12}^2 &= 1 + \cos\theta \\ \rho_{13}^2 &= A\sin^2(\tfrac{1}{2}(\theta + \theta^*)) \\ \rho_{23}^2 &= A\sin^2(\tfrac{1}{2}(\theta - \theta^*)) \end{aligned} \tag{8}$$

where

$$A = \frac{1+m_3}{m_3}.$$

The corresponding second-order Euler-Lagrange equations are:

$$\tag{9} \begin{aligned} \ddot{r} &= -\tfrac{1}{r^2}W + \tfrac{1}{4}r\dot{\theta}^2 \\ \dot{\overline{r^2\dot{\theta}}} &= \tfrac{4}{r}W_\theta. \end{aligned}$$

Next, one can blow-up the triple collision singularity at $r = 0$ by introducing the time rescaling $' = r^{\frac{3}{2}}\,\dot{}$ and the variable $v = r'/r$ [8]. Setting $\tau = \theta'$ gives the following first-order system of differential equations:

$$\tag{10} \begin{aligned} r' &= vr \\ v' &= W(\theta) - \tfrac{1}{2}v^2 + 2rh = \tfrac{1}{4}\tau^2 + \tfrac{1}{2}v^2 - W(\theta) \\ \theta' &= \tau \\ \tau' &= 4\,W_\theta - \tfrac{1}{2}v\tau \end{aligned}$$

with energy equation:

$$\tag{11} \tfrac{1}{2}v^2 + \tfrac{1}{8}\tau^2 - W(\theta) = rh$$

Note that $\{r = 0\}$ is now an invariant set for the flow, called the triple collision manifold. The differential equations are still singular due to the double collisions at $\theta = \pi, \pm\theta^*$. Since m_3 is between m_1 and m_2, only the double collisions at $\theta = \pm\theta^*$ will be relevant. The final coordinate change will eliminate these singularities.

Define new variables u, γ such that

$$\theta = \theta^* \sin u \qquad \gamma = \tau \cos u.$$

Note that $-\theta^* \leq \theta \leq \theta^*$ corresponds to $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$. After calculating the differential equations for u, γ, introduce a further rescaling of time by multiplying the vectorfield by $\theta^* \cos^2 u$. Retaining the prime to denote differentiation with respect to the new time variable one finds

$$\tag{12} \begin{aligned} r' &= \theta^* vr\cos^2 u \\ v' &= \theta^*(G(u) - \tfrac{1}{2}v^2\cos^2 u - 2r\cos^2 u) = \theta^*(\tfrac{1}{4}\gamma^2 + \tfrac{1}{2}v^2\cos^2 u - G(u)) \\ u' &= \gamma \\ \gamma' &= 4\,G_u - \tfrac{1}{2}\theta^* v\gamma\cos^2 u + 4\sin u\cos u(v^2 + 2r) \end{aligned}$$

where

$$\tag{13} \begin{aligned} G(u) &= \cos^2 u\; W(u) \\ W(u) &= \frac{1}{\rho_{12}} + \frac{m_3}{\rho_{13}} + \frac{m_3}{\rho_{23}} \\ \rho_{12}^2 &= 1 + \cos(\theta^* \sin u) \\ \rho_{13}^2 &= A\sin^2(\theta^* \sin^2(\tfrac{1}{2}(u + \tfrac{\pi}{2}))) \\ \rho_{23}^2 &= A\sin^2(\theta^* \sin^2(\tfrac{1}{2}(u - \tfrac{\pi}{2}))). \end{aligned}$$

In deriving the differential equations, the energy relation

$$\tag{14} \tfrac{1}{2}v^2\cos^2 u + \tfrac{1}{8}\gamma^2 - G(u) = rh\cos^2 u$$

has been used. It has been assumed that $h = -1$. Because of the scaling symmetry, this is no loss of generality.

The Newtonian potential function $W(u)$ has singularities at $u = \pm\frac{\pi}{2}$ corresponding to the double collision singularities (see figure 2). One can show that $W(u)$ is

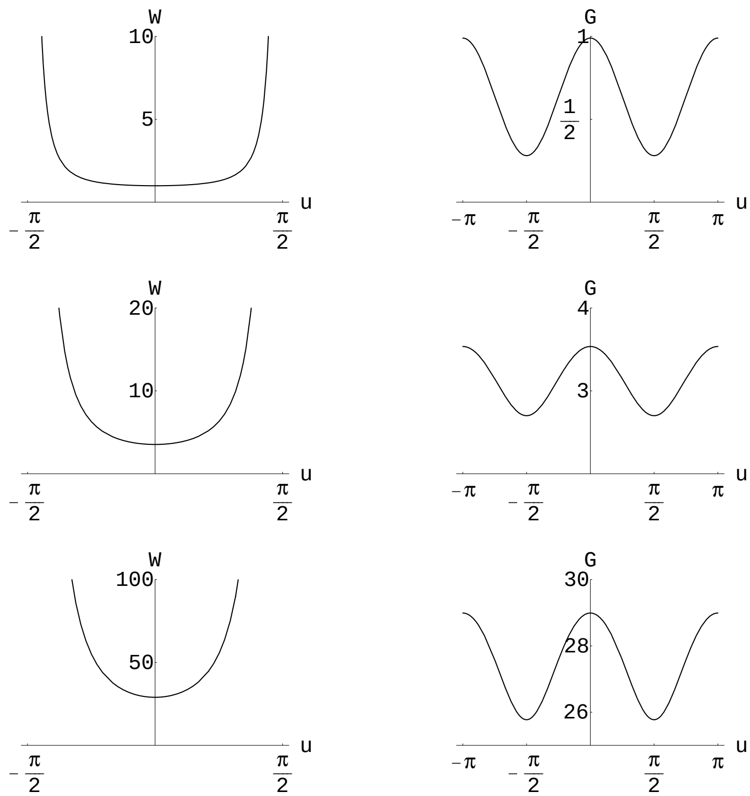

FIGURE 2. Newtonian potential $W(u)$ and regularized potential $G(u)$ for $m_3 = \frac{1}{10}, 1, 10$

convex on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with second derivative $W_{uu}(u) > 0$. There is a unique critical point at $u = 0$ corresponding to Euler's collinear central configuration. On the interval $(0, \frac{\pi}{2})$, $W_u(u) > 0$. The regularized potential function $G(u)$, on the other hand, extends analytically to the double collisions to give an analytic function on the entire real line of period π. $G(u)$ has critical points at $u = 0, \frac{\pi}{2}$ (and their translates by integer multiples of π) with $u = 0$ giving its maximum.

The differential equations (12) represent the collinear three-body problem for configurations with m_3 in the middle, with triple collision blown-up and double collisions regularized. The variable u describing the shape need not be confined to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. As u ranges over the real axis the shape variable θ oscillates between the double collisions at $\pm\theta^*$ and the mass m_3 bounces back and forth between m_1 and m_2.

The proof presented below makes use of the symmetry of the regularized potential function $G(u)$ to construct a symmetrical periodic solution as in figure 1. Figure 3 shows the periodic orbit with $m_3 = 1$ in the (u, r) and (u, v) planes. The first quarter of the orbit is a segment running from the vertical line $u = 0$ to the line $u = \frac{\pi}{2}$ with $v = 0$ at both endpoints. Then, using the symmetry of the vectorfield, it follows that one can piece together the rest of the orbit by reflection through the

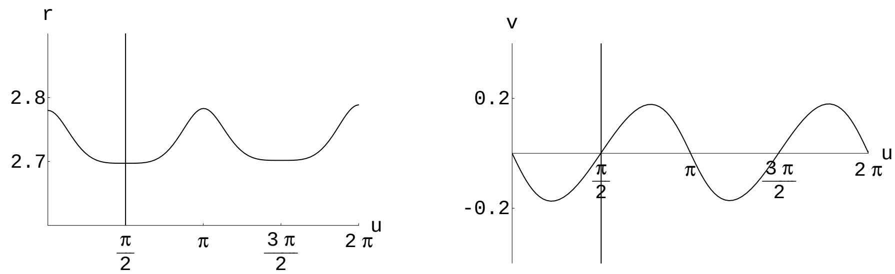

FIGURE 3. Symmetrical periodic solution viewed in the (u, r) and (u, v)-planes. The orbit can be reconstructed from the segment between $u = 0$ and $u = \frac{\pi}{2}$ by reflection and translation.

line $u = \frac{\pi}{2}$ followed by horizontal translation by π. The existence of the required segment will be proved by a topological shooting argument.

3. A Wazewski Set

Consider the system (12) on the manifold of fixed energy $h = -1$. The goal of this section is to construct a Wazewski set for the flow on this three-dimensional manifold.

A Wazewski set for a flow $\phi_t(x)$ on a topological space X is a subset $\mathcal{W} \subset X$ satisfying technical hypotheses which guarantee that the time required to exit $\mathcal{W}$ depends continuously on initial conditions [11, 2]. To formulate these hypotheses, let $\mathcal{W}^0$ be the set of points in $\mathcal{W}$ which eventually leave $\mathcal{W}$ in forward time, and let $\mathcal{E}$ the set of points which exit immediately:

$$\mathcal{W}^0 = \{x \in \mathcal{W} : \exists t > 0, \phi_t(x) \notin \mathcal{W}\}$$
$$\mathcal{E} = \{x \in \mathcal{W} : \forall t > 0, \phi_{[0,t)}(x) \not\subset \mathcal{W}\}.$$

Clearly, $\mathcal{E} \subset \mathcal{W}^0$. Given $x \in \mathcal{W}^0$ define the *exit time*

$$\tau(x) = \sup\{t \geq 0 : \phi_{[0,t)} \subset \mathcal{W}\}.$$

Note that $\tau(x) = 0$ if and only if $x \in \mathcal{E}$. The appropriate hypotheses which guarantee continuity of τ are [2]:

a. If $x \in \mathcal{W}$ and $\phi_{[0,t]} \subset \overline{\mathcal{W}}$, then $\phi_{[0,t]} \subset \mathcal{W}$

b. $\mathcal{E}$ is a relatively closed subset of $\mathcal{W}^0$

The choice of the set $\mathcal{W}$ is motivated by the shooting argument outlined at the end of the last section. Let

$$\mathcal{W} = \{(r, v, u, \gamma) : (14) \text{ holds}, r \geq 0, v \leq 0, 0 \leq u \leq \tfrac{\pi}{2}, \gamma \geq 0\}.$$

To visualize $\mathcal{W}$ one can use coordinates (r, v, u) on the energy manifold, using the energy equation to find γ. The energy manifold projects to the solid region

$$r + \tfrac{1}{2}v^2 \leq W(u). \tag{15}$$

Then $\mathcal{W}$ appears as in figure 4. The upper surface in the figure, where equality holds in (15) corresponds to $\gamma = 0$. The figure shows several other important features which will be explained in due course.

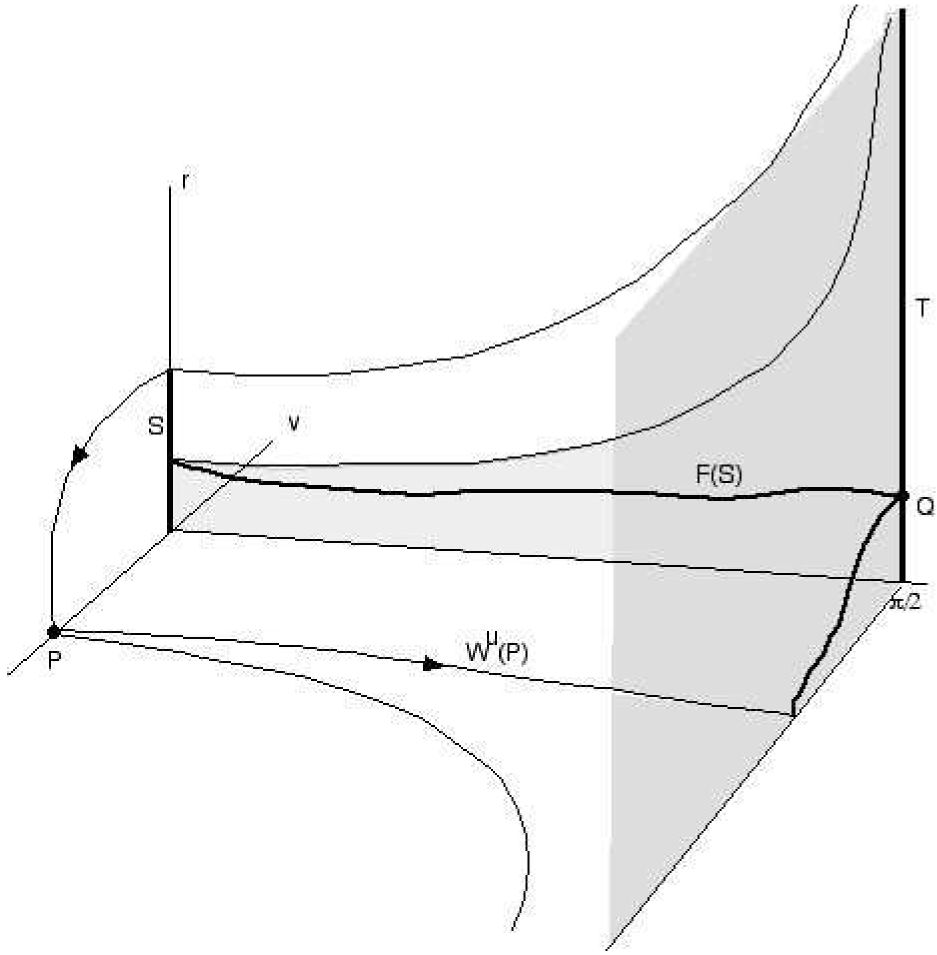

FIGURE 4. The Wazewski set $\mathcal{W}$ and a sketch of the existence proof .

Note that for any orbit segment in $\mathcal{W}$ the projection to the (u, r)-plane will lie in the vertical strip over $[0, \frac{\pi}{2}]$ (see figure 3). Moreover, the curve will move in a south-easterly direction, with u non-decreasing and r non-increasing.

The rest of this section is devoted to proving:

Theorem 1. *$\mathcal{W}$ is a Wazewski set for the flow on the energy manifold.*

The verification of defining property **a** is immediate since $\mathcal{W}$ is a closed subset of $\mathbf{R}^4$. To check property **b**, one must first identify the subsets $\mathcal{W}^0, \mathcal{E}$. It turns out that every solution beginning in $\mathcal{W}$ eventually leaves, with one exception — the Eulerian triple collision solution mentioned in the introduction. Recall that this is the simple homothetic solution obtained by putting m_3 at the midpoint of m_1, m_2 and releasing the masses with zero initial velocities.

To describe it more precisely, first note that there is a unique equilibrium point in $\mathcal{W}$ at $P = (r, v, u, \gamma) = (0, -v_0, 0, 0)$ where $v_0 = \sqrt{2G(0)}$. This corresponds to triple collision with limiting shape $u = 0$, the Eulerian central configuration. As shown by McGehee [8], the equilibrium point P is hyperbolic with two-dimensional stable manifold and one-dimensional unstable manifold. Indeed, using the coordinates (v, u, γ) as local coordinates, one finds that the linearized differential equations at the restpoint have matrix

$$\begin{bmatrix} -\theta^* v_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 4(G_{uu}(0) + v_0^2) & \frac{1}{2}\theta^* v_0 \end{bmatrix}.$$

Using the fact that $v_0 > 0$ and $G_{uu}(0) + v_0^2 = G_{uu}(0) + 2G(0) = W_{uu}(0) > 0$ one finds that the eigenvalues are $\lambda_1 = -\theta^* v_0 < 0$ and two other real eigenvalues $\lambda_2 < 0, \lambda_3 > 0$. The eigenvectors are $(1, 0, 0), (0, 1, \lambda_2), (0, 1, \lambda_3)$ respectively. The first stable eigenvector is tangent to the Eulerian homothetic orbit $\mathcal{H} = \{(r, v, u, \gamma) : u = \gamma = 0\}$. This orbit forms one of the edges of the Wazewski set $\mathcal{W}$ (the curved edge with the arrow in figure 4). Note that the other stable eigenvector $(0, 1, \lambda_2)$ points out of $\mathcal{W}$ since any scalar multiple of it which has $u > 0$ must have $\gamma < 0$. It follows that $\mathcal{H} \cap \mathcal{W} = W^s(P) \cap \mathcal{W}$.

It is clear that any initial condition in $\mathcal{H} \cap \mathcal{W}$ remains there for all $t \geq 0$ and converges to P. However, these are the only orbits which remain in $\mathcal{W}$:

Lemma 1. *$\mathcal{W}^0 = \mathcal{W} \setminus \mathcal{H}$.*

Proof. Let $x_0 = (r_0, v_0, u_0, \gamma_0) \in \mathcal{W}$. As long as the corresponding solution $\phi_t(x_0)$ remains in $\mathcal{W}$, one has bounds $0 \leq r(t) \leq r_0$ and $0 \leq u(t) \leq \frac{\pi}{2}$. The energy equation shows that $\gamma(t)^2 \leq 8G(u(t))$ so $\gamma(t)$ is also bounded above and below. There is an upper bound $v(t) \leq 0$ but the energy equation does not provide a fixed lower bound. There is a lower bound on the derivative, however:

$$v' \geq -\theta^* G(u).$$

Then it follows from standard existence theorems that $\phi_t(x_0)$ continues to exist as long as it remains in $\mathcal{W}$.

Now suppose $x_0 \in \mathcal{W} \setminus \mathcal{H}$. It will be shown that $\phi_t(x_0)$ eventually leaves $\mathcal{W}$. If $u_0 = 0$ then $u'(0) = \gamma_0 > 0$ since $x_0 \notin \mathcal{H}$. It follows that for every $t_0 > 0$, $u(t_0) > 0$. Thus it suffices to consider initial conditions with $u_0 > 0$.

Fix any $u_0 > 0$ and let $\mathcal{W}_{u_0} = \{x \in \mathcal{W} : u \geq u_0\}$. Note that since $u(t)$ is non-decreasing along orbits in $\mathcal{W}$, $\mathcal{W}_{u_0}$ is positively invariant relative to $\mathcal{W}$. It will be shown below that there exist constants $c_0 > 0$ and $d_0 > 0$ such that for every $x \in \mathcal{W}_{u_0}$ either $\gamma \geq c_0$ or $\gamma' \geq d_0$. This will be enough to show that $\phi_t(x_0)$ must eventually leave $\mathcal{W}$. To see this, write $\mathcal{W}_{u_0} = \mathcal{W}^+_{u_0} \cup \mathcal{W}^-_{u_0}$ where $\mathcal{W}^+_{u_0} = \{x \in \mathcal{W}_{u_0} : \gamma \geq c_0\}$ and $\mathcal{W}^-_{u_0} = \{x \in \mathcal{W}_{u_0} : 0 \leq \gamma \leq c_0\}$. Then since $\gamma' \geq d_0 > 0$ in $\mathcal{W}^-_{u_0}$, it follows that an orbit segment can remain there for time c_0/d_0, at most. Furthermore $\mathcal{W}^+_{u_0}$ is positively invariant relative to $\mathcal{W}_{u_0}$. Finally, an orbit can remain in $\mathcal{W}^+_{u_0}$ for at most time $\pi/(2c_0)$ since $u' = \gamma \geq c_0$. Thus every orbit beginning in $\mathcal{W}_{u_0}$ eventually leaves $\mathcal{W}$, as required.

It remains to construct $c_0 > 0, d_0 > 0$ such that either $\gamma \geq c_0$ or $\gamma' \geq d_0$ for all $x \in \mathcal{W}_{u_0}$. First note that for $u = \frac{\pi}{2}$, the energy equation gives $\gamma^2 = 8G(\frac{\pi}{2}) > 0$. If the constant c_0 is chosen less than $\sqrt{8G(\frac{\pi}{2})}$ then $\gamma \geq c_0$ will hold for $u = \frac{\pi}{2}$. On the other hand, if $u_0 \leq u < \frac{\pi}{2}$, the equation for γ' can be written

$$\gamma' = 4G_u(u) - \tfrac{1}{2}\theta^* v\gamma \cos^2 u + \tan u(8G(u) - \gamma^2) \geq 4G_u(u) + \tan u(8G(u) - \gamma^2)$$

since $v \leq 0$ and $\gamma \geq 0$ in $\mathcal{W}$. Now

$$4G_u(u)\cot u + 8G(u) = \frac{4\cos^3 u}{\sin u} W_u(u) > 0$$

on $[u_0, \frac{\pi}{2}]$. Let $k > 0$ be its minimum value on this interval. Then one has

$$\gamma' \geq \tan u(k - \gamma^2) \geq \tan u_0(k - \gamma^2).$$

Taking $c_0 = \min(\sqrt{k/2}, \sqrt{8G(\frac{\pi}{2})})$ and $d_0 = k \tan u_0/2$ completes the proof. QED

To find the immediate exit set $\mathcal{E}$ one must examine the boundary points of $\mathcal{W}$ (see figure 4). It is convenient to distinguish two subsets of the boundary. Let $x = (r, v, u, \gamma)$ and let

$$\mathcal{B}_1 = \{x \in \mathcal{W} : u = \tfrac{\pi}{2}\}$$
$$\mathcal{B}_2 = \{x \in \mathcal{W} : v = 0, 2r\cos^2 u \leq G(u)\}.$$

Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ are closed subsets of $\mathcal{W}$ (shaded in figure 4). Therefore the following lemma completes the verification of hypothesis **b** and the proof the theorem 1.

Lemma 2. *The immediate exit set of $\mathcal{W}$ is $\mathcal{E} = \mathcal{B}_1 \cup \mathcal{B}_2$.*

Proof. Let $x \in \mathcal{B}_1$. Since $u = \frac{\pi}{2}$ and $u' = \gamma = \sqrt{8G(\frac{\pi}{2})} > 0$, x is clearly an immediate exit point. Thus $\mathcal{B}_1 \subset \mathcal{E}$.

Next consider a point in $x \in \mathcal{B}_2$. Points with $u = \frac{\pi}{2}$ are already known to be in $\mathcal{E}$, so one may assume $0 \leq u < \frac{\pi}{2}$. By definition, $v = 0$ and so one has

$$v' = \theta^*(G(u) - 2r\cos^2 u) \geq 0.$$

If $2r\cos^2 u < G(u)$ then $v' > 0$ and x is an immediate exit point. On the other hand if $2r\cos^2 u = G(u)$, one has $v = v' = 0$ and one finds that the second derivative reduces to

$$v'' = \theta^*\gamma(G_u(u) + 4r\cos u \sin u).$$

Furthermore the energy equation simplifies to $\gamma^2 = 4G(u) > 0$. Now the expression in parentheses is positive unless $u = 0$. Therefore if $2r\cos^2 u = G(u)$ and $0 < u < \frac{\pi}{2}$, one has $v = v' = 0$ and $v'' > 0$ and again x is an immediate exit point. Finally if $u = v = 0$ and $2r\cos^2 u = G(u)$ one has $v = v' = v'' = 0$. The third derivative is found to be

$$v''' = \theta^*(G_{uu}(0) + 2G(0)) = \theta^* W_{uu}(0) > 0.$$

Again, x is an immediate exit point and $\mathcal{B}_2 \subset \mathcal{E}$.

To complete the proof, it remains to show that there are no other immediate exit points. Suppose, for the sake of contradiction, that $x_0 \in \mathcal{W}$ is an immediate exit point which is not in $\mathcal{B}_1 \cup \mathcal{B}_2$. Rather than considering all of the various faces, edges, and corners of the boundary it is easier to consider the logically possible ways that x_0 could exit and to rule them out.

First, one could exit by having $r_0 = 0$ but $r(t) < 0$ for small positive times. However, this is impossible because $\{r = 0\}$ is the invariant triple collision manifold.

Next, one might have $u_0 = 0$ and $u(t) < 0$ for small positive times. Clearly this requires $u'(0) = \gamma_0 \leq 0$ and since $x_0 \in \mathcal{W}$ this means $\gamma_0 = 0$. But $u_0 = \gamma_0 = 0$ defines Euler's orbit $\mathcal{H}$ and points of $\mathcal{H}$ are certainly not immediate exit points.

A third possibility is that $v_0 = 0$ and then $v(t)$ increases to become positive. This forces $v' = \theta^*(G(u) - 2r\cos^2 u) \geq 0$ and so $x_0 \in \mathcal{B}_2$.

The last possibility is $\gamma_0 = 0$ with $\gamma(t)$ becoming negative. Since $u_0 = \gamma_0 = 0$ defines $\mathcal{H}$ and $u_0 = \frac{\pi}{2}$ represents $\mathcal{B}_1$, one may assume $0 < u_0 < \frac{\pi}{2}$. In this case, it follows from the proof of lemma 1 that there are positive constants c_0, d_0 such that $\gamma' \geq d_0 > 0$ whenever $\gamma < c_0$. In particular, this applies to points with $\gamma_0 = 0$ and so this mode of exiting is also impossible. This completes the proof. QED

4. The Shooting Argument

To construct symmetric periodic orbits, it suffices to show that there is an initial condition with $u = v = 0$ which can be followed across $\mathcal{W}$ to an exit state with $u = \frac{\pi}{2}, v = 0$. Let

$$\mathcal{S} = \{(r, v, u, \gamma) \in \mathcal{W} : u = v = 0\}$$
$$\mathcal{T} = \{(r, v, u, \gamma) \in \mathcal{W} : u = \phi/2, v = 0\}.$$

$\mathcal{S}$ and $\mathcal{T}$ are two of the edges in the boundary of the three-dimensional Wazewski set $\mathcal{W}$ (shown as bold vertical lines in figure 4). Along $\mathcal{S}$ one has $0 \leq r \leq G(0)$ and $\gamma = \sqrt{8(W(0) - r)}$. Viewing r as a parameter along $\mathcal{S}$ one sees that the endpoint with $r = 0$ lies in the triple collision manifold while the endpoint with $r = W(0)$ is a point of the Eulerian homothetic orbit $\mathcal{H}$. Let

$$\mathcal{S}_0 = \{(r, v, u, \gamma) \in \mathcal{W} : u = v = 0, 0 \leq r < G(0)\}.$$

Then $\mathcal{S}_0 \subset \mathcal{W}_0$, that is, all of these points eventually exit $\mathcal{W}$ through $\mathcal{E}$. Since $\mathcal{W}$ is a Wazewski set, the time required to reach $\mathcal{E}$ depends continuously on initial conditions and so there is a continuous flow-defined map $F : \mathcal{S}_0 \to \mathcal{E}$. Now the target set $\mathcal{T}$ is contained in $\mathcal{E}$. It remains to show that $F(\mathcal{S}_0) \cap \mathcal{T} \neq \emptyset$.

Again taking r as a parameter along $\mathcal{S}_0$, one sees that the part of $\mathcal{S}_0$ with $0 \leq r \leq G(0)/2$ is contained in $\mathcal{B}_2 \subset \mathcal{E}$. These points exit $\mathcal{W}$ immediately and so the map F is the identity there. On the other hand, points of $\mathcal{S}_0$ with $r > G(0)/2$ will enter the interior of $\mathcal{W}$ and emerge elsewhere. The proof will be completed by studying the behavior of points near the other end of $\mathcal{S}_0$, that is, with $r \approx G(0)$. By continuity of the flow these points will follow the Eulerian homothetic orbit $\mathcal{H}$ down to a neighborhood of the Eulerian equilibrium point at $P = (0, -v_0, 0, 0)$. It follows from the lambda lemma that they will then follow a branch of $W^u(P)$.

Now $W^u(P)$ is a one-dimensional manifold contained entirely in the invariant triple collision manifold $r = 0$. One of the two branches is contained in $\mathcal{W}$. In fact it is contained in $\mathcal{W}_0$ and so can be followed under the flow to the exit set $\mathcal{E}$. The next lemma describes where it exits:

Lemma 3. *The branch of $W^u(P)$ in $\mathcal{W}$ exits $\mathcal{W}$ at a point of the form $(0, v, \frac{\pi}{2}, \gamma)$ with $v < 0$.*

In other words, having started with $u \approx 0$ and $v \approx -v_0 < 0$ it reaches the double collision at $u = \frac{\pi}{2}$ before it reaches $v = 0$.

Using this lemma, one can conclude the shooting argument as follows. In the last section it was shown that the immediate exit set consists of two pieces $\mathcal{B}_1$ and $\mathcal{B}_2$ of the boundary of $\mathcal{W}$. As shown in figure 4, $\mathcal{B}_1$ and $\mathcal{B}_2$ are two-dimensional surfaces meeting along the edge $\mathcal{T}$. The continuous map F takes points of $\mathcal{S}_0$ near $r = 0$ to $\mathcal{B}_2 \setminus \mathcal{T}$ and points near $r = G(0)$ to $\mathcal{B}_1 \setminus \mathcal{T}$. It follows that there must exist at least one intersection point $Q \in F(\mathcal{S}_0) \cap \mathcal{T}$. This completes the existence proof for the symmetric periodic orbits.

Theorem 2. *Let three positive masses $m_1 = m_2$ and m_3 and a negative energy h be given. Then there exists a symmetric periodic solution of the collinear three-body problem with energy h and regularized double collisions of the following type. During the first quarter period, the masses move from the Eulerian central configuration with m_3 at the midpoint of m_1, m_2 to a double collision between m_2 and m_3. At the moment of double collsion the velocity of m_1 is zero. The second quarter of the orbit is the time-reverse of the first and the second half is the reflection of the first half with the roles of m_1 and m_2 reversed.*

The proof also shows that the moment of intertia I is decreasing on the first quarter of the orbit, after which it increases, decreases and increases again during the other quarter periods.

It only remains to prove lemma 3 about the branch of the unstable manifold $W^u(P)$.

Proof of Lemma 3. Consider the differential equations for u, v. Using the energy relation and the fact that the unstable branch lies in the triple collision manifold, $r = 0$, these can be written:

$$v' = \theta^*(G - \tfrac{1}{2}v^2\cos^2 u)$$
$$u' = \gamma$$

where $\gamma^2 = 8(G(u) - \frac{1}{2}v^2\cos^2 u)$. So, parametrizing the branch by the variable u, one has

$$\frac{dv}{du} = \frac{\theta^*}{4}\sqrt{2G(u) - v^2\cos^2 u} \leq \frac{\theta^*}{4}\sqrt{2G(u)}.$$

Now $G(u)$ is decreasing for $0 \leq u \leq \frac{\pi}{2}$ (see figure 2). Hence $G(u) \leq G(0)$ and

$$\frac{dv}{du} \leq \frac{\theta^*}{4}\sqrt{2G(0)} = \frac{\theta^*\, v_0}{4}$$

where $v_0 = \sqrt{2G(0)}$. It follows that the change in v for $0 \leq u \leq \frac{\pi}{2}$ satisfies:

$$\Delta v \leq \frac{\theta^*\pi v_0}{8} \leq \frac{\pi^2 v_0}{16} < v_0$$

since $\theta^* \leq \frac{\pi}{2}$. Since the branch of $W^u(P)$ begins near P where $u = 0$ and $v = -v_0$, it follows that it arrives at $u = \frac{\pi}{2}$ before reaching $v = 0$ as claimed. QED

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