FREE TIME MINIMIZERS FOR THE THREE-BODY PROBLEM

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Abstract. Free time minimizers of the action (called“semi-static” solutions by Mañe) play a central role in the theory of weak KAM solutions to the Hamilton-Jacobi equation [12]. We prove that any solution to Newton's three-body problem which is asymptotic to Lagrange's parabolic homothetic solution is eventually a free time minimizer. Conversely, we prove that every free time minimizer tends to Lagrange's solution, provided the mass ratios lie in a certain large open set of mass ratios. We were inspired by the work of [8] which showed that every free time minimizer for the N-body problem is parabolic, and therefore must be asymptotic to the set of central configurations. We exclude being asymptotic to Euler's central configurations by a second variation argument. Central configurations correspond to rest points for the McGehee blown-up dynamics. The large open set of mass ratios are those for which the linearized dynamics at each Euler rest point has a complex eigenvalue.

1. Introduction

Solutions of the Newtonian N-body problem can be characterized as extremals of the action functional. This paper is about special solutions which minimize the action in a strong sense.

A configuration of N point masses in $\mathbb{R}^d$ is given by $q = (q_1, \ldots, q_N) \in (\mathbb{R}^d)^N$. The *negative* of the Newtonian potential is the positive function

$$U(q) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$$

where $r_{ij} = |q_i - q_j|$ are the interparticle distances and $m_i > 0$ are the masses. The action of an absolutely continuous curve $\gamma : [a,b] \to (\mathbb{R}^d)^N$ is

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))dt$$

where $L : (\mathbb{R}^d)^{2N} \to]0,\infty]$ is the Lagrangian

$$L(q,v) = K(v) + U(q) = \frac{1}{2}\sum_{i=1}^N m_i |v_i|^2 + U(q)$$

with $v_i = \dot q_i$ being the velocity of the ith mass.

Newton's equations are the Euler-Lagrange equations for this action. This means that if $\gamma(t)$ is a solution of the N-body problem defined on a time interval J and if $[a,b] \subset J$, then the restriction $\gamma|_{[a,b]}$ is an extremal of the action among curves which connect $p = \gamma(a)$ to $q = \gamma(b)$ in the same time interval. A *free time minimizer* will be a curve for which each $\gamma|_{[a,b]}$ is not only an extremal, but an action minimizer, even among curves connecting p to q over a different time interval.

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More precisely, denoting $\mathbb{E} = (\mathbb{R}^d)^N$ let $\mathcal{C}(p, q, \tau)$ be the set of absolutely continuous curves connecting two given configurations $p, q \in \mathbb{E}$ in time $\tau > 0$:

$$\mathcal{C}(p, q, \tau) = \{\gamma : [a, b] \to \mathbb{E} \text{ absolutely continuous } : b - a = \tau,\ \gamma(a) = p, \gamma(b) = q\}.$$

Let $\mathcal{C}(p, q)$ denote the set of absolutely continuous curves connecting two configurations $p, q \in \mathbb{E}$ with no time restriction:

$$\mathcal{C}(p, q) = \bigcup_{\tau > 0} \mathcal{C}(p, q, \tau)\ .$$

Define the finite time action function $h : \mathbb{E} \times \mathbb{E} \times (0, +\infty) \to \mathbb{R}$ by

$$h(p, q; \tau) = \inf\{A(\gamma) | \gamma \in \mathcal{C}(p, q, \tau)\},$$

and the *Mañé critical action potential* by $S : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$,

$$S(p, q) = \inf\{A(\gamma) | \gamma \in \mathcal{C}(p, q)\} = \inf\{h(p, q; \tau) | \tau > 0\}.$$

The infimum defining h is achieved for every pair of configurations $p, q \in \mathbb{E}$. The infimum defining S is achieved if and only if $p \neq q$. These facts are essentially due to the lower semicontinuity of the action.

Definition 1. *A* free time minimizer *(FTM) defined on an interval $J \subset \mathbb{R}$ is an absolutely continuous curve $\gamma : J \to \mathbb{E}$ which satisfies $A(\gamma|_{[a,b]}) = S(\gamma(a), \gamma(b))$ for all compact subinterval $[a, b] \subset J$.*

It is natural to wonder what are the FTMs for the Newtonian N-body problem and there have been several recent papers devoted to this question, including works by Da Luz-Maderna [8], Maderna-Venturelli [19] , Maderna [17, 18], Percino-Sanchez Morgado [31], and Yu-Hu [37].

For $d \geq 2$, a simple example of such a solution for $N = 3$ is the parabolic, homothetic solution associated to the equilateral central configuration (see Figure 1). Recall that a *central configuration* is a configuration $c \in \mathbb{E}$ which yields homothetic motions $\gamma(t) = \lambda(t)c$ to Newton's equations, where $\lambda(t)$ is a scalar function. It is well-known that central configurations can be characterized as critical points of the restrictions of the Newtonian potential to the level sets of the moment of inertia $I = \sum m_i |q_i|^2$. If c is such a constrained critical point then it gives rise to a homothetic solution where the function $\lambda(t)$ satisfies the one-dimensional Kepler equation $\ddot{\lambda}\lambda^2 = -U(c)/I(c)$. But only the zero-energy solution $\lambda(t) = \alpha t^{2/3}$ where $\alpha^3 = 9U(c)/2I(c)$ gives rise to a FTM.

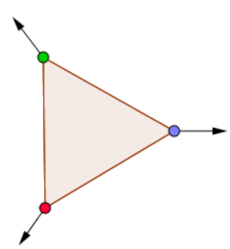

FIGURE 1. The Lagrange parabolic homothetic solution: an equilateral triangle expanding at the rate $t^{2/3}$.

We define a *minimal configuration* to be a central configuration c which minimizes the restriction of U to its moment of inertia level. For the three-body problem, the minimal configurations are exactly the equilateral triangles. Maderna and Venturelli [19] proved that if c is a minimal configuration of the N-body problem, then the corresponding zero-energy homothetic solution γ_c is a free time minimizer.

As time $t \to \infty$ in $\gamma_c(t) = \alpha t^{2/3} c$ the velocity of each body tends to zero. We will take this property as the definition of a *parabolic solution.* Parabolic solution have many other special properties. For example, they have energy zero and all of the mutual distances tend to infinity with asymptotic rate $t^{\frac{2}{3}}$. Da Luz and Maderna [8] proved that every FTM is a parabolic solution.

The qualitative study of parabolic solutions goes at least back to Chazy [3], [4, chapter III] who took the $t^{\frac{2}{3}}$ asymptotics as his definition. For $N = 3$, he showed that any parabolic solution $\gamma(t)$ asymptotes to γ_c for some central configuration c, in the sense that

$$\lim_{t\to\infty} t^{-2/3}\gamma(t) = c. \tag{1}$$

See subsection 3.4 for a proof of an equivalent statement. In that section we make use of a modern approach to the study of parabolic solutions based on a variation of McGehee's blow-up method [23, 24]. This method was introduced to study orbits near triple collision. There is a close relationship between triple collision orbits and parabolic orbits already noted by Chazy. For example, the homothetic orbits $\gamma_c(t)$ begin at triple collision when $t = 0$ and then tend parabolically to infinity. By blowing up the coordinates and slowing down the timescale, McGehee was able to add a "collision manifold" as a boundary to the ordinary phase space. Orbits ending in triple collision are now asymptotic to a restpoint in the collision manifold. Each central configuration determines two restpoints, one serving as a limit for triple collisions in forward time and one for collisions in backward time. By a similar method, one can add a boundary manifold "at infinity" such that parabolic solutions converge to restpoints.

For the three-body problem, the central configurations are the Lagrange equilateral triangle configurations and three types of collinear central configurations discovered by Euler, one for each choice of which body lies between the other two. These results beg us to ask three questions. Are the Euler parabolic homothetic solutions FTMs? To which of the five types of central configurations may a FTM be asymptotic? Among the parabolic solutions, which ones are FTMs ?

For the three-body problem, the works [18, 19, 8, 31] show that given any equilateral central configuration c and any initial configuration q there is an FTM beginning at q and converging asymptotically to c. In other words, among the parabolic orbits converging to c there are many FTM solutions, enough so that one can find one with an arbitrary initial configuration. If l is one of the restpoints at infinity corresponding to a Lagrange configuration, let $W^s_+(l)$ denote the part of its stable manifold with $r < \infty$. The existence theorem in [31] shows that there is a part of $W^s_+(l)$ consisting entirely of FTM solutions which projects onto the whole configuration space. Our main result is that this "minimizing" part of the stable manifold contains a whole neighborhood of l in $W^s(l)$. Since every parabolic solution eventually enters such a neighborhood we have

Theorem 1. *Every parabolic solution $\gamma(t), 0 \le t < \infty$ of the three-body problem which is asymptotic to a Lagrange configuration is a free time minimizer upon restriction to a sub-interval $T \le t < \infty$, T large enough.*

Our second result rules out the Euler configurations as possible asymptotic limits of FTMs, at least for a large range of masses which we call the "spiraling range", depicted in figure 2.

Theorem 2. *In the spiraling range of mass ratios, every free time minimizer $\gamma(t)$ for the three-body problem is asymptotic to a Lagrange configuration: the limit c of equation (1) is an equilateral triangle. Equivalently, these orbits lie in the stable manifold of one of the Lagrange restpoints at infinity, as described in subsection 3.4.*

The spiraling range refers to the occurrence of certain non-real eigenvalues which make an appearance in [3] where Chazy studied both triple collision and parabolic solutions. Both have

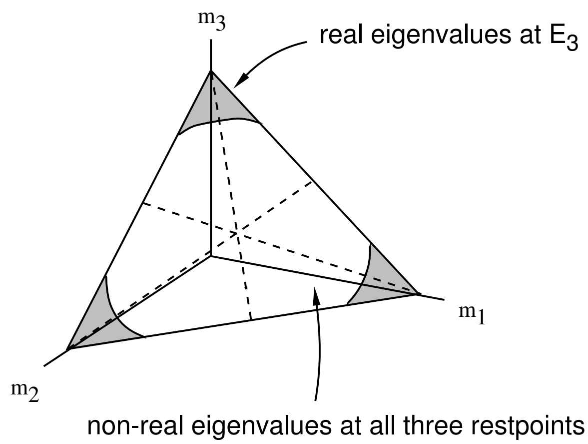

FIGURE 2. The spiraling range of mass ratios, depicted in the mass simplex $m_1+m_2+m_3=1$. If E_i denotes the collinear central configuration with mass m_i in the middle, then the mass for which spiraling does not occur occupies the small shaded region near the corresponding vertex, a region in which where m_i is much larger than the other masses. The spiraling range, where all three central configurations have nonreal eigenvalues, is represented by the large unshaded region of the simplex. The figure is based on results in [33].

asymptotic growth rate $t^{\frac{2}{3}}$ with triple collision corresponding to $t \to 0$ and parabolic motion to $t \to \infty$. Chazy found that the next term in the series expansions of solutions are of order $t^{\frac{2}{3}+p}$ where p is a certain characteristic root. These characteristic roots are proportional to the eigenvalues discussed below. In sections 10 and 11 of [3], Chazy gives equations for the roots for the three-body problem and notes that non-real roots can occur for parabolic motion near the Eulerian configurations. These same eigenvalues were found and explicit formulas given by Siegel in his study of the triple collision singularity [34, 35]. In the modern approach based on McGehee blow-up, triple collision orbits and parabolic orbits make up the stable and unstable manifolds of restpoints in the collision manifold or the manifold at infinity. The eigenvalues of these restpoints are the same, up to a real positive scaling, as those found by Chazy and Siegel. Moreover the eigenvalues within the collision manifold are the same as those within the manifold at infinity for corresponding restpoints.

Definition 2.

(i) We call a central configuration spiraling *if the linearized flow at the associated restpoint on the collision manifold and the manifold at infinity have a non-real eigenvalue.*

(ii) If each of the three Euler configurations is spiraling then we say that the mass ratios $[m_1 : m_2 : m_3]$ *are in the* spiraling range*.*

The spiraling range of mass ratios forms a rather large open connected set. (See figure 2.). It can be characterized explicitly by algebraic inequalities as follows. Consider one of the Euler configurations, say the configuration s_2 for which m_2 lies between m_1 and m_3. It turns out that distance ratio $r = r_{23}/r_{12}$ for this configuration is the positive root to the fifth degree equation

$$(m_2+m_3)r^5 + (2m_2+3m_3)r^4 + (m_2+3m_3)r^3 \\ - (3m_1+m_2)r^2 - (3m_1+2m_2)r - (m_1+m_2) = 0. \tag{2}$$

(This root is unique by Descartes' rule of signs.) Let

$$\nu = \frac{m_1(1+3r+3r^2)+m_3(3r^3+3r^4+r^5)}{(m_1+m_3)r^2+m_2(1+r)^2(1+r^2)}. \tag{3}$$

In Proposition 4 we show that spiraling occurs if and only if

$$\nu > \frac{1}{8}. \tag{4}$$

Similar inequalities at the other two Euler restpoints are obtained by permuting the masses and together they define the spiraling range.

It is clear from figure 2 that for fixed mass ratios, the spiraling condition can only be violated at one of the Euler restpoints and this happens only when the middle mass is much larger than the other two masses. If $\nu \leq \frac{1}{8}$ at one of the Euler points our proof, based on local variations, does not produce nearby curves with lower action. But being a FTM is a strong, global condition and it is an open question whether the corresponding Euler homothetic motion is an FTM or not.

Remark. Following McGehee's work, several authors, including one of us, have used the oscillatory phenomenon near triple collision to prove existence of interesting near-collision orbits [9, 36, 25, 26, 27, 28]. In these it was discovered that if the mass ratios are in the spiraling range then certain complicated chaotic behaviors, described by symbolic dynamics, are guaranteed to occur for the corresponding three-body problem.

2. Outline of Proofs and some further results

2.1. **Theorem 1.** A great part of our work here was motivated by the construction of a weak KAM solution in [31], which is also central to the proof of Theorem 1. The FTM solutions built in [31] are found as curves "calibrated " by a weak KAM solution of the Hamilton-Jacobi equation.

Definition 3. *A* viscosity *or* weak KAM *solution of the Hamilton-Jacobi equation*

$$\|Du(x)\|^2 = 2U(x). \tag{5}$$

is a function $u : \mathbb{E} \to \mathbb{R}$ that satisfies:

(a) u is dominated, *i.e. $u(q) - u(p) \leq S(p,q)$ for all $p, q \in \mathbb{E}$.*
(b) For any $q \in \mathbb{E}$ there is an absolutely continuous curve $\alpha : [0,\infty) \to \mathbb{E}$ such that $\alpha(0) = q$ and α is calibrated *by u, meaning that $u(\alpha(t)) - u(q) = A(\alpha|_{[0,t]})$ for any $t > 0$.*

Notice that a calibrated curve is a FTM.

To build weak KAM solutions one mimics a Busemann construction in Riemannian geometry

Definition 4. *The Busemann function B_γ associated to the free time minimizer $\gamma : [0,+\infty) \to \mathbb{E}$ is*

$$B_\gamma(x) = \sup_{t>0}\left[S(0,\gamma(t)) - S(x,\gamma(t))\right] = \lim_{t\to+\infty}\left[S(0,\gamma(t)) - S(x,\gamma(t))\right].$$

In fact, Percino and Sanchez were able to prove

Proposition 1. [31] *If c is a minimal configuration then the Busemann function associated to the parabolic homothetic motion γ_c is a weak KAM solution. Moreover, for any configuration q, the corresponding calibrated curve will be asymptotic to c.*

The key lemma for Theorem 1 is

Lemma 1. *The stable manifold $W^s_+(l)$ of a Lagrange restpoint c at infinity is an immersed Lagrangian submanifold and the local stable manifold $W^s_{loc}(l)$ is an exact Lagrangian graph, i.e. there is a smooth $f : V \subset \mathbb{E} \to \mathbb{R}$ such that*

$$W^s_{loc}(l) \cap (V \times \mathbb{E}^*) = \text{graph}\, df.$$

Remark. McGehee blow-up yields a 2:1 correspondence between normalized central configurations c and rest points of the Newtonian flow, continued to infinity. See section 3.1. Using these coordinates it makes sense to talk about neighborhoods of normalized configurations at infinity. See definition 6.

The basic idea of the proof of Theorem 1 is as follows. Where smooth, the graph of the differential of the weak KAM solution constructed by Percino and Sanchez [31] forms a Lagrangian manifold. By lemma 1 $W^s_+(l)$ is also a Lagrangian submanifold associated to c. The main point of the proof of theorem 2 is that these two Lagrangian submanifolds agree near infinity. The following corollary of lemma 2 is part of the argument and is also interesting in its own right

Corollary 1. *Let B_c be the Buseman function associated to a homothetic parabolic Lagrange solution γ_c. Then there is a $T > 0$ and a truncated conical neighborhood V of $\gamma_c([T, \infty))$ (see figure 3) on which B_c is smooth and such that*

$$\{(\gamma(t), \dot\gamma(t)^*) : \gamma \textit{ curve calibrated by } B_c, \gamma(0) \in V\} = W^s_{loc}(l) \cap (V \times \mathbb{E}^*).$$

Remark. In this last set-theoretic equality, $\dot\gamma(t)^*$ denotes the momentum, or covector, dual to the velocity $\dot\gamma(t)$ by way of the Legendre transform. The conical neighborhood $V \subset \mathbb{E}$ corresponds to a neighborhood of infinity as per definition 6. The Newtonian flow, with center of mass set to zero, occurs on the phase space $\mathbb{E} \times \mathbb{E}^*$. See the beginning of the next section.

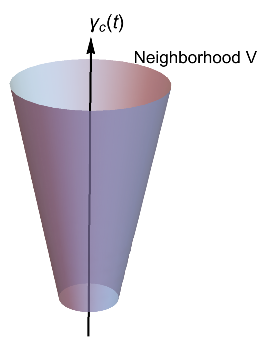

FIGURE 3. A conical neighborhood of a parabolic homothetic solution.

2.2. **Theorem 2.** In trying to understand free time minimizers, it is natural to consider the second variation of the action along a solution segment. If this second variation is negative then we call the corresponding solution "variationally unstable". The action of a variationally unstable solution can be decreased by deforming the solution curve in the direction of the negative variation, so such solution curves cannot be free-time minimizers. Thus, Theorem 2 follows immediately from

Lemma 2. *(A) The Euler parabolic homothety solution is variationally unstable if and only if the mass ratio associated to that Euler central configuration is spiraling (definition 2) .*

(B) For these same mass ratios, any parabolic solution asymptotic to that Euler parabolic homothety solution is variationally unstable.

Remark. In the language of McGehee blow-up, part (B) concerns solutions in the stable manifold of the corresponding Eulerian restpoint at infinity.

Remark. Barutello and Secchi[1] established a significant part of (A) as well as part (B) of lemma 2 in their proposition 4.3, namely they established the variational instability of solutions asymptotic to Euler in the spiraling range. We learned from one of the referes that after this manuscript was submitted, Barutello, Hu, Portaluri and Terracini posted a preprint [2] where the variational (2nd order) stability outside that range is proved.

Note that the variational stability of an Euler solution when there is no spiraling does not imply that these solutions are FTMs. This remains an open problem.

3. Restpoints at infinity, eigenvalues and parabolic motions

In this section we develop the McGehee blow-up method for parabolic orbit and compute the eigenvalues needed for the proofs of the main theorems.

The configuration space of the three-body problem is three copies of the Euclidean space $\mathbb{R}^d$ and we assume that $d \geq 2$. By a standard trick from introductory physics we can reduce the number of degrees of freedom by d by insisting that the center of mass be zero:

$$m_1 q_1 + m_2 q_2 + m_3 q_3 = 0 \tag{6}$$

and correspondingly that the total linear momentum be zero:

$$m_1 v_1 + m_2 v_2 + m_3 v_3 = 0. \tag{7}$$

Write $\mathbb{E} \cong \mathbb{R}^{2d} \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ for either $2d$-dimensional linear subspace, so that, upon making these restrictions, the phase space for Newton's equation is $\mathbb{E} \times \mathbb{E}$.

To write Newton's equations as Hamilton's equations we turn velocities v into momenta p with the help of the masses. We do this using the "mass metric" which can be written $\langle v, w\rangle_m = v \cdot Mw$ where $M = \mathrm{diag}(m_1, \ldots, m_1, m_2, \ldots, m_2, m_3, \ldots, m_3)$ is the mass matrix. The mass matrix turns velocities v into covectors (momenta) $p = Mv$ and, upon restriction to $\mathbb{E}$, provides us with an identification $\mathbb{E} \cong \mathbb{E}^*$.

Newton's equations read

$$\ddot{q} = \nabla_m U(q) \tag{8}$$

where the gradient is with respect to the mass-metric: $dU(q)(v) = \langle \nabla_m U(q), v\rangle_m$. The energy $H : \mathbb{E} \times \mathbb{E}^* \to \mathbb{R}$ is the function

$$H(q, p) = K(M^{-1}p) - U(q)$$

and is constant along solutions to Newton's equations. The origin in $\mathbb{E}$ represents triple collision and distance from the origin with respect to the mass metric is:

$$r(q) = \sqrt{I(q)} = \sqrt{\langle q, q\rangle_m}.$$

Each configuration $q \in \mathbb{E}$ determines a unique *normalized configuration* $s = q/r$ with $I(s) = r(s)^2 = 1$. A *central configuration or CC* is a point $q \in \mathbb{E}$ such that

$$\nabla_m U(q) + \lambda q = 0 \text{ for some } \lambda \in \mathbb{R}. \tag{9}$$

Using the homogeneity of the potential it is easy to see that $\lambda = \frac{U(q)}{r(q)^2}$ which reduces to $\lambda = U(c)$ for a normalized central configuration.

3.1. **McGehee coordinates and blow-up.** Following McGehee, define new variables

$$r = \sqrt{\langle q, q\rangle_m}, \qquad s = \frac{q}{r}, \qquad z = r^{\frac{1}{2}}\dot{q}.$$

Introduce the function

$$v = \langle z, s\rangle_m. \tag{10}$$

Define the new time variable τ by $\dfrac{d}{d\tau} = r^{\frac{3}{2}}\dfrac{d}{dt}$ and write $f' = \dfrac{df}{d\tau}$. Then Newton's equations 8 when written in the new, blown-up variables reads

$$r' = vr \tag{11}$$

$$s' = z - vs \tag{12}$$

$$z' = \nabla_m U(s) + \frac{1}{2}vz. \tag{13}$$

In deriving these use the homogeneity of the potential to see that $U(q) = r^{-1}U(s)$ and $\nabla_m U(q) = r^{-2}\nabla_m U(s)$. Observe that the equations now make sense when $r = 0$. Since $r = 0$ corresponds to triple collision we say that the triple collision singularity has been blown-up into the invariant manifold $\{r = 0\}$. Also observe that for all values of r the differential equations for (s, z) are independent of r.

The new blown up equations make sense on $\mathbb{R} \times \mathbb{R}^{3d} \times \mathbb{R}^{3d}$. Their evolution respects the center of mass, linear momentum, and normalization constraints

$$m_1 s_1 + m_2 s_2 + m_3 s_3 = 0, \qquad m_1 z_1 + m_2 z_2 + m_3 z_3 = 0 \qquad \text{and } \langle s, s\rangle_m = 1. \tag{14}$$

We work with these constraints imposed, so that our phase space is the $4d$-dimensional:

$$X = [0, \infty) \times S^{2d-1} \times \mathbb{E} \subset \mathbb{R} \times \mathbb{R}^{3d} \times \mathbb{R}^{3d} = \mathbb{R}^{6d+1}.$$

The large linear embedding space $\mathbb{R}^{6d+1}$ will be useful later on when linearizing the flow.

The flow still preserves the energy levels $H(q, p) = h$ which are now given by

$$H(s, z) = \frac{1}{2}\langle z, z\rangle_m - U(s) = rh.$$

We will be especially interested in the case $h = 0$.

The rate of change of $v = \langle z, s\rangle_m$ satisfies

$$v' = \langle z, z\rangle_m - U(s) - \frac{1}{2}v^2 = \frac{1}{2}\langle z, z\rangle_m - \frac{1}{2}v^2 + rh.$$

If $r = 0$ (triple collision) or $h = 0$ (zero energy), this simplies to

$$v' = \frac{1}{2}(\langle z, z\rangle_m - v^2) \geq 0$$

where the nonnegativity follows from the Cauchy-Schwarz inequality. Thus v is a Lyapunov function on the triple collision ($r = 0$) and zero energy ($h = 0$) submanifolds.

When $h = 0$ we study the motions with $r \to \infty$ by replacing r by $u = r^{-1}$. Then equation (11) is replaced by:

$$u' = -vu \tag{15}$$

while equations (12), (13) and the energy equation $H(s, z) = 0$ remain unchanged. Now $\{u = 0\}$ is invariant and represents the dynamics at infinity for the zero energy problem. Our zero-energy phase space is now the $4d - 1$-dimensional submanifold $H(s, z) = 0$ of $[0, \infty] \times S^{2d-1} \times \mathbb{E}$ where $r = \infty$ corresponding to $u = 0$.

3.2. **Restpoints.** We first consider the planar three-body case and then indicate what are the changes for $d > 2$. A point (s, z) is an equilibrium point for the differential equations (12), (13) if and only if

$$v^2 = 2U(s) \qquad z = vs$$

and

$$\nabla_m U(s) + U(s)s = 0 \tag{16}$$

which is exactly the equation (9) for a normalized central configuration, with $\lambda = U(s)$.

Equation (16) can also be viewed in another way. The normalization condition $\langle s, s\rangle_m = 1$ defines a three-sphere $\mathcal{E} \subset \mathbb{E}$. Then (16) is the equation for critical points of the restriction of $U(s)$ to this sphere. In fact, the equations can be written $\tilde{\nabla} U(s) = 0$ where

$$\tilde{\nabla} U(s) = \nabla_m U(s) + U(s)s. \tag{17}$$

is the gradient of the restriction of U to the three-sphere with respect to the metric on this sphere induced by the mass metric. (The restricted gradient of a general smooth $f : \mathbb{E} \to \mathbb{R}$ is $\tilde{\nabla} f(s) = (\nabla_m f(s))^T$, where v^T is the orthogonal projection of the vector $v \in \mathbb{E}$ to the tangent space at s to the sphere. Using Euler's identity for homogeneous functions, we find that if f is homogeneous of degree α then $(\tilde{\nabla} f(s)) = \nabla_m f - \alpha f(s)s$, hence the expression for $\tilde{\nabla} U$.) Because of the rotational symmetry of the potential, there are actually five manifolds of critical points, one for each of the five central configuration shapes.

Each normalized central configuration s_0 determines two equilibrium points in the triple collision manifold $(0, s_0, z_0)$ where

$$z_0 = v_0 s_0 \qquad v_0 = \pm\sqrt{2U(s_0)}. \tag{18}$$

For the zero energy problem we also get two equilibrium points at infinity with the same (s_0, z_0) and $u = 0$.

For a given normalized central configuration c, its equilibria at collision and infinity are connected by the zero energy parabolic homothetic orbit $\gamma_c(t)$. These are precisely the zero energy solutions such that r changes while the (s, z) remains at their equilibrium values (18). The size r is given by

$$r(\tau) = \exp(v_0\tau) \qquad \text{or} \qquad u(\tau) = \exp(-v_0\tau).$$

Here τ denotes the normalized time variable and $v_0 = \langle s_0, z_0\rangle$.

3.3. **Stable and unstable manifolds.** As just discussed the parabolic solutions lie in the stable manifold of one of the rest points at infinity. For the three-body problem, these rest points are all hyperbolic (after allowing for rotational symmetry) and their Lyapunov exponents will play an important role in what follows. These exponents have been calculated before [34, 35] but we will present the results here (with some details relegated to an Appendix) for the sake of completeness and to correct some unfortunate typos which appeared in [28].

Consider the variational equation of the blown-up differential equations (11), (12), (13) at one of the equilibrium points $p = (0, s, z)$. Differentiation and evaluation at $r = 0$ gives the 13×13 matrix:

$$A = \begin{bmatrix} v & 0 & 0 \\ 0 & -vI - sz^tM & I - ss^tM \\ 0 & D\nabla_m U(s) + \frac{1}{2}zz^tM & \frac{1}{2}vI + \frac{1}{2}zs^tM \end{bmatrix}.$$

For a restpoint at infinity, that is to say $u = 0$, the only difference in A is that the upper left v of the matrix becomes $-v$ so the two cases can be considered together.

Some words may be helpful regarding the terms sz^tM, ss^tM, zz^tM and zs^tM. The term sz^tM in the 2-2 block of A, for example, describes the linear operator taking δs to $s\langle z, \delta s\rangle_m$. All these terms arise from linearizing the quadratic function $v = \langle s, z\rangle_m = s^tMz$ which occurs as a factor in equations (12), (13) , and (11).

Let $(\delta r, \delta s, \delta z) \in T_pX \subset \mathbb{R}^{13}$ denote a tangent vector to X at p, where X is our eight-dimensional phase space defined by the normalization equations (14). Linearizing the first and last of the normalization equations we find

$$m_1\delta s_1 + m_2\delta s_2 + m_3\delta s_3 = 0 \text{ and } s^tM\delta s = 0 \tag{19}$$

which defines the three-dimensional tangent space to the sphere $\mathcal{E} \subset \mathbb{E}$ at s. Since $z = vs$ we also have

$$z^tM\delta s = 0$$

so we can ignore the terms involving z^tM in the second column of A. In general, it is not true that $s^tM\delta z = 0$, however, this equality does hold for vectors lying in T_pX and also tangent to the energy manifold $H(s, z) = 0$, since for such vectors

$$\delta H = z^tM\delta z - \nabla U(s)\cdot\delta s = vs^tM\delta z + U(s)\, s^tM\delta s = vs^tM\delta z.$$

Thus vectors with $\delta H = 0$ also have $s^tM\delta z = 0$ and for these the third column of A also simplifies.

One easily checks that the vectors $(\delta r, \delta s, \delta z) = (1, 0, 0)$ and $(\delta r, \delta s, \delta z) = (0, 0, s)$ are eigenvectors in T_pX with eigenvalues $\lambda_1 = v$ and $\lambda_2 = v$. The first vector satisfies $\delta H = 0$ while the second vector satisfies $\delta H = v \neq 0$. The subspace $\delta r = \delta H = 0$ is a 6-dimensional subspace of T_pX invariant under A. It follows that the other 6 eigenvectors of A restricted to T_pX must lie in this subspace. Dropping the z^tM and s^tM terms from A we find that the other eigenvectors are of the form $(0, \delta s, \delta z)$ where $(\delta s, \delta z)$ is an eigenvector of the 12×12 matrix

$$B = \begin{bmatrix} -vI & I \\ D\nabla_m U(s) & \frac{1}{2}vI \end{bmatrix}$$

The following lemma (see [10] for the lemma's origin) gives the eigenvectors and eigenvalues of B in terms of those of $D\nabla_m U(s)$ or equivalently those of [1]

$$D\tilde{\nabla}U(s) = D\nabla_m U(s) + U(s)I. \tag{20}$$

Lemma 3. *Let s be a normalized central configuration and $z = vs$ where $v^2 = 2U(s)$. If a vector δs satisfying (19) is an eigenvector of $D\tilde{\nabla}U(s)$ with eigenvalue α then the vectors $(\delta s, k_\pm\delta s)$ are eigenvectors of B with eigenvalues*

$$\lambda_\pm = \frac{-v \pm \sqrt{v^2 + 16\alpha}}{4} \qquad k_\pm = v + \lambda_\pm$$

Proof. The assumptions imply that

$$D\nabla_m U(s)\delta s = (\alpha - U(s))\delta s = (\alpha - \frac{1}{2}v^2)\delta s.$$

Then

$$B(\delta s, k\delta s)^t = ((-v + k)\delta s, (\alpha - \frac{1}{2}v^2 + \frac{1}{2}vk)\delta s)^t.$$

Setting this equal to $\lambda(\delta s, k\delta s)^t$ leads to the equations

$$k = v + \lambda \qquad \lambda^2 + \frac{1}{2}v\lambda - \alpha = 0$$

and solving for λ, k completes the proof. □

[1] From equation (17) we get that $D\tilde{\nabla}U$ equals the expression of equation (20) plus the term $\nabla U(s) \otimes s^tM$ which we ignore since $\langle s, \delta s\rangle_m = 0$.

It is easy to guess several eigenvectors of $D\tilde{\nabla}U(s)$. First we have the configuration vector $s = (s_1, s_2, s_3)$. Since U is homogeneous of degree -1, $\nabla_m U$ is homogeneous of degree -2 and therefore $D\nabla_m U(s)s = -2\nabla_m U(s)$. Since s is a central configuration, we have $\nabla_m U(s)+U(s)s = 0$ and therefore $D\tilde{\nabla}_m U(s)s = 2U(s)s$. In other words, s is an eigenvector with eigenvalue $\alpha = 2U(s)$. However, s does not satisfy the normalization conditions (19). Next, consider translation vectors of the form $w = (k, k, k)$ where $k \in \mathbb{R}^2$. Since $\nabla_m U(s)$ is translation invariant, we have $D\nabla_m(s)w = 0$ and $D\tilde{\nabla}_m(s)w = U(s)w$. There are two independent eigenvectors of this type with $\alpha = U(s)$, but these also do not satisfy (19).

Finally, the rotational symmetry of $U(s)$ implies that $s^\perp$ is an eigenvector with eigenvalue $\alpha = 0$ where $s^\perp = (s_1^\perp, s_2^\perp, s_3^\perp)$, the vector with each s_i rotated by 90° in the plane. $s^\perp$ does satisfy the normalization conditions. Lemma 3 gives eigenvalues $\lambda = -\frac{v}{2}, 0$ for B. There are two more eigenvectors satisfying (19) and they will determine what we will call the nontrivial eigenvalues of $D\tilde{\nabla}U(s)$. These will be calculated in the appendix. For now, we just record the results.

Proposition 2. *Let s be a normalized central configuration and $p = (r, s, z) = (0, s, vs)$ one of the triple collision restpoints where either $v = \sqrt{2U(s)}$ or $v = -\sqrt{2U(s)}$. Let α_1, α_2 be the two nontrivial eigenvalues of $D\tilde{\nabla}U(s)$. Then the eight Lyapunov exponents of the variational equations on T_pX are*

$$\lambda = v, v, -\frac{v}{2}, 0, \frac{-v \pm \sqrt{v^2 + 16\alpha_1}}{4}, \frac{-v \pm \sqrt{v^2 + 16\alpha_2}}{4}.$$

The eigenvalues for an equilibrium at infinity $p = (u, s, z) = (0, s, vMs)$ are the same except the first one becomes $-v$.

The Lagrangian (equilateral) critical points form circles of local minima in $\mathcal{E}$. The corresponding nontrivial eigenvalues are both positive.

Proposition 3. *The nontrivial eigenvalues of $D\tilde{\nabla}U(s)$ at an equilateral central configuration are*

$$\alpha_1, \alpha_2 = \frac{3U(s)}{2}\left(1 \pm \sqrt{k}\right)$$

where

$$k = \frac{(m_1 - m_2)^2 + (m_1 - m_3)^2 + (m_2 - m_3)^2}{2(m_1 + m_2 + m_2)^2}.$$

The four corresponding nontrivial eigenvalues at one of the Lagrangian equilibrium points at triple collision or at infinity are

$$\lambda = \frac{-v}{4}\left(1 \pm \sqrt{13 \pm 12\sqrt{k}}\right)$$

After allowing for the rotation, the Eulerian, collinear critical points are saddles with one positive and one negative nontrivial eigenvalue. Their values depend on the shape of the configuration. Consider the collinear central configuration with m_2 between m_1 and m_3. Instead of normalizing the configuration we can look for critical points of the translation and scale invariant function

$$F(s) = I(s)U(s)^2$$

with no constraints. It is easy to see that if s is a critical point of F then the corresponding normalized configuration satisfies (16). Using F, we may assume without loss of generality that

$$s_1 = (0, 0) \qquad s_2 = (r, 0) \qquad s_3 = (1 + r, 0)$$

where $0 < r < 1$. This gives a function of one variable $F(r)$ and setting $F'(r) = 0$ leads to the fifth degree equation (2).

Proposition 4. *The nontrivial eigenvalues of $D\tilde{\nabla}U(s)$ at the collinear central configuration with m_2 between m_1, m_3 are*

$$\alpha_1, \alpha_2 = -U(s)\nu,\, U(s)(3+2\nu)$$

with ν given by (3). The four corresponding nontrivial eigenvalues at the Eulerian equilibrium points at triple collision or at infinity are

$$\lambda = \frac{-v}{4}\left(1 \pm \sqrt{1-8\nu}\right), \frac{-v}{4}\left(1 \pm \sqrt{25+16\nu}\right). \tag{21}$$

The values at the other Eulerian restpoints are found by permuting the subscripts on the masses.

Note that the Eulerian restpoints have a pair of nonreal eigenvalues if and only if $\nu > \frac{1}{8}$. Figure 2 shows the masses for which $\nu > \frac{1}{8}$ for each of the three Eulerian restpoints.

We close this section with some remarks about dimensions $d > 2$. The crucial matrix $D\tilde{\nabla}U(s)$ will be $3d \times 3d$ and the subspace determined by the normalization conditions (19) will have dimension $2d-1$ and we have an action of the rotation group $\mathbf{SO}(d)$.

For the equilateral central configurations, the $\mathbf{SO}(d)$ orbit is the unit tangent bundle of the sphere $\mathbb{S}^{d-1}$. To see this, choose an orthonormal frame e_1, e_2 in the plane of the triangle. A point of the orbit is given by specifying the images u_1, u_2. The vector u_1 lies on the sphere $\mathbb{S}^{d-1}$ and u_2 is a unit vector tangent to $\mathbb{S}^{d-1}$ at u_1. It follows that $D\tilde{\nabla}U(s)$ has $2d-3$ independent eigenvectors with eigenvalue $\alpha = 0$. Together with the two nontrivial eigenvalues from proposition 3, this account for all $2d-1$ eigenvalues satisfying (19). If we ignore the normalization we also have the eigenvector s with eigenvalue $2U(s)$ and d translation eigenvectors with eigenvalue $U(s)$.

For the collinear central configurations, the $\mathbf{SO}(d)$ orbit is the sphere $\mathbb{S}^{d-1}$ since a point of the orbit is determined by specifying the image of the line containing the bodies. It follows that $D\tilde{\nabla}U(s)$ has $d-1$ independent eigenvectors with eigenvalue $\alpha = 0$. If we fix the line of the bodies, there will be a $d-1$ independent planes in which we can select an eigenvector with eigenvalue $\alpha = -\nu U(s)$ as in proposition 4. Hence the $2d-1$ normalized eigenvalues of $D\tilde{\nabla}U(s)$ consist of the positive nontrivial eigenvalue $U(s)(3+2\nu)$ together with $d-1$ copies of 0 and $d-1$ copies of $-\nu U(s)$.

Using lemma 3 we can work out the eigenvalues of the restpoints at infinity for the three-body problem in $\mathbb{R}^d$. Each Lagrangian restpoint l is part of a $T_1\mathbb{S}^{d-1}$ of restpoints. The eigenvalues of l are the same as in proposition 3 except that the eigenvalues $\lambda = 0, -v/2$ are repeated $2d-3$ times. There will be a total of $2d$ stable eigenvalues in the normalized phase space of dimension $4d$.

Each Eulerian restpoint e is part of a $\mathbb{S}^{d-1}$ of restpoints. The eigenvalues of e are the same as in proposition 3 except that the eigenvalues $\lambda = 0, -v/2, \frac{-v}{4}\left(1 \pm \sqrt{1-8\nu}\right)$ are each repeated $d-1$ times.

3.4. **Parabolic motions tend to rest points at infinity.** The qualitative study of parabolic solutions goes at least back to Chazy [3, 4]. For Chazy, parabolic solutions were those such that the mutual distances tend to infinity as $t^{\frac{2}{3}}$ as for the parabolic solutions of the two-body problem. Chazy derived many other properties and characterizations of these solutions, some of which have been adopted as definitions by later authors. For the sake of clarity we are going to choose a simple definition and derive the properties relevant to our needs in this subsection.

Definition 5. *A solution to the three-body problem is (future) parabolic if the solution's domain contains a positive half line $[t_0, \infty)$ and if the Newtonian velocities of all three bodies tend to zero as (Newtonian) time tends to infinity.*

Remark. Define "past parabolic" by letting time tend to negative infinity. We stick with future parabolic for simplicity.

Proposition 5. *Any parabolic solution has energy* 0 *and lies in the stable manifold of one of the rest points* $(u, s, z) = (0, s, vs)$ *at infinity. Conversely every solution in the stable manifold of a rest point at infinity and such that* $u > 0$ *is a parabolic solution.*

Proof. For the three-body problem, it seems that most of this result follows from Chazy's work [3, 4]. But he did not use our definition of parabolic and, of course, he did not refer to stable manifolds at infinity. For completeness we will give a proof here using ideas from [22, 6]. We denote velocity by y so as not to confuse it with the McGehee variable v.

For a parabolic motion, the kinetic energy $K(y) \to 0$ as $t \to \infty$. The energy equation $K(y) - U(q) = h$ and the fact that $U(q) > 0$ imply that $h \le 0$. To rule out the case $h < 0$ we use the Lagrange-Jacobi identity $\ddot{I}(t) = 2K + 2h$. If $K \to 0$ and $h < 0$ then $\ddot{I}(t)$ has a negative upper bound for t sufficiently large which forces $I(t) \to 0$ (total collapse) in finite time. Such a solution would not exist for large $t > 0$.

It is also easy to see that $r(t) \to \infty$ as $t \to \infty$. Indeed the energy equation gives

$$r(t)K(y) = r(t)U(q) = U(s).$$

Now the normalized potential $U(s)$ has a positive lower bound depending only on the choice of masses, so $K(y) \to 0$ implies $r(t) \to \infty$.

The main theorem in Marchal and Saari [22] describes the asymptotic behavior as $t \to \infty$ for any solution of the n-body problem which exists for all $t \ge 0$. For any such solution, either $r(t)/t \to \infty$ or else all of the position vectors satisfy $q_k = A_k t + O(t^{\frac{2}{3}})$ for some constant vectors A_k, possibly zero. The second case implies that either $r(t)/t \to L$ for some $L > 0$ or else $r(t) = O(t^{\frac{2}{3}})$ (the latter holding when all $A_k = 0$). We will show that in fact we have $r(t) = O(t^{\frac{2}{3}})$ for parabolic orbits. To see this note that given any $\epsilon > 0$ there is t_0 such that $\ddot{I}(t) = 2K < \epsilon$ for $t \ge t_0$. Then

$$I(t) \le I(t_0) + \dot{I}(t_0)(t - t_0) + \frac{1}{2}\epsilon(t - t_0)^2.$$

If $r(t)/t \to L \in (0, \infty]$ then $I(t)/t^2 \to L^2 \in (0, \infty]$ and we have the contradiction that $0 < L^2 < \frac{1}{2}\epsilon$ for all $\epsilon > 0$.

Next we show that the McGehee variable $v(t)$ tends to a finite limit $v(t) \to \bar{v} > 0$ as $t \to \infty$. Recall that $v(t)$ is non-decreasing since v is a Liapanov function on the zero energy surface. Also

$$v(t) = r^{-1}r'(t) = \sqrt{r(t)}\,\dot{r}(t) = \frac{2}{3}\frac{d}{dt}r(t)^{\frac{3}{2}}.$$

Since $r(t) \to \infty$ we have $v(t) > 0$ for t large so either $v(t)$ approaches some $\bar{v} > 0$ or else $v(t) \to \infty$. But integration gives

$$\frac{1}{t}\left(\frac{2}{3}r(t)^{\frac{3}{2}} - \frac{2}{3}r(0)^{\frac{3}{2}}\right) = \frac{1}{t}\int_0^t v(s)\,ds.$$

If $v(t) \to \infty$ as $t \to \infty$ we would get $r(t)^{\frac{3}{2}}/t \to \infty$ contradicting $r(t) = O(t^{\frac{2}{3}})$.

Finally we can use the dynamics on the infinity manifold to finish the proof. First note that the estimate $r(t) = O(t^{\frac{2}{3}})$ shows that the rescaled time τ with $\dot{\tau}(t) = r^{\frac{3}{2}}(t)$ satisfies $\tau(t) \to \infty$ as $t \to \infty$, so a forward parabolic orbit exists for all $\tau \ge 0$ and we have $u(\tau) = 1/r(\tau) \to 0$ as $\tau \to \infty$. We claim that the ω limit set of our parabolic orbit consists of one of the restpoints in the manifold $\{u = 0\}$.

Consider the subset

$$S = \{(u, s, z) : u = 0, H(s, z) = 0, v = \bar{v}\}$$

where $\bar{v}$ is the limit of $v(\tau)$ for a certain parabolic solution. We will show that this solution avoids a neighborhood of the double collision singularities in S. Since the solution exists for all $t \ge 0$ and $\tau \ge 0$ it does not actually have a double collision, but we want to avoid a whole neighborhood.

Let $w = s' = z - vs$, the component of z tangent to the ellipsoid $\mathcal{E}$. The energy equation can be written $\frac{1}{2}v^2 + \frac{1}{2}|w|^2 = U(s)$. Since $U(s) \to \infty$ at collision while v^2 is bounded near S, it follows that $|w|$ is large near collision. Now

$$v' = \frac{1}{2}(\langle z, z\rangle_m - v^2) = \frac{1}{2}|w|^2$$

while the arclength σ in $\mathcal{E}$ satisfies $\sigma' = |s'| = |w|$. Hence the rate of change of v with respect to arclength is $\frac{1}{2}|w|$. From this we see that there is some neighborhood $\mathcal{U}$ of the double collision singularities in S such that any initial condition in $\mathcal{U}$ crosses into the set $v > \bar{v}$. So our solution avoids $\mathcal{U}$.

We conclude that our parabolic solution must converge to the compact set $S' = S \setminus \mathcal{U}$ as $\tau \to \infty$. Therefore it has a nonempty, compact ω limit set contained in S'. For orbits in the limit set we must have $v'(\tau) = \frac{1}{2}|w(\tau)|^2 = 0$ for all τ and this happens only at the restpoints, that is, at the points $(u, s, z) = (0, c, vc)$ where c is a central configuration. For the three-body problem, the restpoints form five manifolds and the eigenvalue computations show that these are normally hyperbolic invariant manifolds. It follows that the omega limit set consists of just one of the restpoints.

The converse is easier. Assume that $r(\tau) \to \infty$ and $(s(\tau), z(\tau)) \to (c, \bar{v}c)$ where $\bar{v} = \sqrt{2U(c)}$. Since $K(z) = \sqrt{r}K(y) \to U(s)$ we have $K(y) \to 0$ as $\tau \to \infty$. Inverting the change of timescale we find that $t \to \infty$ as $\tau \to \infty$ so the solution is parabolic.

□

4. Proof of Lemma 2

In this section we revert to the classical variables and timescale but will make use of the eigenvalue computation of proposition 4.

4.1. **Proof of part (A) of Lemma 2.**

Proof. Consider perturbations γ^ϵ of a homothetic parabolic motion $\gamma_c(t)$ associated to a central configuration c with $r(c) = 1$, so $\gamma_c(t) = \rho(t)c$ where $\rho(t) = \left(\frac{9}{2}U(c)\right)^{\frac{1}{3}} t^{\frac{2}{3}}$. For any $[a, b] \subset \mathbb{R}^+$, $v \in C^2([a, b], \mathbb{E})$ such that $v(a) = v(b) = 0$, $\langle c, v(t)\rangle_m = 0$, we consider the variation of γ_c of the form

$$\gamma^\epsilon(t) = \rho(t)(c + \epsilon v(t)) \tag{22}$$

so that

$$\frac{d\gamma^\epsilon(t)}{d\epsilon} = \rho(t)v(t), \ \frac{d^2\gamma^\epsilon(t)}{d\epsilon^2}\Big|_{\epsilon=0} = 0$$

$$A(\gamma^\epsilon; a, b) = \frac{1}{2}\int_a^b \langle \dot{\gamma}^\epsilon(t), \dot{\gamma}^\epsilon(t)\rangle_m + \int_a^b U(\gamma^\epsilon(t))\, dt$$

$$\frac{dA(\gamma^\epsilon; a, b)}{d\epsilon} = \int_a^b \langle \dot{\gamma}^\epsilon(t), \frac{d\dot{\gamma}^\epsilon(t)}{d\epsilon}\rangle_m\, dt + \int_a^b \langle \nabla_m U(\gamma^\epsilon(t)), \frac{d\gamma^\epsilon(t)}{d\epsilon}\rangle_m\, dt.$$

We have

$$\frac{d^2A(\gamma^\epsilon;a,b)}{d\epsilon^2}\Big|_{\epsilon=0} = \int_a^b \Big[\langle \frac{d\dot\gamma^\epsilon(t)}{d\epsilon}, \frac{d\dot\gamma^\epsilon(t)}{d\epsilon}\rangle_m + \langle \frac{d\gamma^\epsilon(t)}{d\epsilon}, D\nabla_m U(\gamma^\epsilon(t))\frac{d\gamma^\epsilon(t)}{d\epsilon}\rangle_m\Big]_{\epsilon=0} dt$$

$$= \int_a^b \Big[\rho(t)^2\langle \dot v(t),\dot v(t)\rangle_m + 2\rho(t)\dot\rho(t)\langle v(t),\dot v(t)\rangle_m + \dot\rho(t)^2\langle v(t),v(t)\rangle_m\Big] dt$$

$$+ \int_a^b \rho(t)^{-1}\langle v(t), D\nabla_m U(c)v(t)\rangle_m \, dt$$

where in the last line we used that $D\nabla_m U$ is homogeneous of degree -3. We remark that the quantity $\langle v, D\nabla_m U(c)v\rangle_m$ occuring in the last term is just the Hessian of U at c, evaluated at the vector v. See the remark at the end of this subsection. Integrate by parts and use that $\ddot\rho = -U(c)/\rho^2$ to get

$$\int_a^b \Big[2\rho(t)\dot\rho(t)\langle v(t)\dot v(t)\rangle_m + \dot\rho(t)^2\langle v(t),v(t)\rangle_m\Big] dt = \int_a^b \dot\rho(t)\frac{d}{dt}(\rho(t)\langle v(t),v(t)\rangle_m)dt$$

$$= -\int_a^b \ddot\rho(t)\rho(t)\langle v(t),v(t)\rangle_m dt = \int_a^b \rho(t)^{-1}U(c)\langle v(t),v(t)\rangle_m dt,$$

so that

$$\frac{d^2A(\gamma^\epsilon;a,b)}{d\epsilon^2}\Big|_{\epsilon=0} = \int_a^b \rho(t)^2\langle \dot v(t),\dot v(t)\rangle_m \, dt + \int_a^b \rho(t)^{-1}[\langle v(t), D\tilde\nabla U(c)v(t)\rangle_m] \, dt. \tag{23}$$

where $D\tilde\nabla U(c) = D\nabla U(c) + U(c)I$. This is exactly the quantity (20) which occurred in the computation of the eigenvalues in the last section.

Now recall from proposition 4 that $D\tilde\nabla U(c)$ has an eigenvector, say $\delta s = z$ with a negative eigenvalue $\alpha_1 = -U(c)\nu$. Take $v(t) = \varphi(t)z$ for the variation of equation (22) where $\varphi \in C^2([a,b])$ with $\varphi(a) = \varphi(b) = 0$. Plugging into equation (23) we find that

$$\frac{d^2A(\gamma^\epsilon;a,b)}{d\epsilon^2}\Big|_{\epsilon=0} = Q(\varphi;a,b) = \int_a^b \Big[\rho(t)^2\dot\varphi(t)^2 + \alpha_1\rho(t)^{-1}\varphi(t)^2\Big] dt. \tag{24}$$

This quadratic form in φ is positive definite i.e. $Q(\varphi;a,b) \geq 0$, for any $[a,b] \subset (0,\infty)$ if and only if its Euler-Lagrange equation

$$(\rho(t)^2y')' - \alpha_1\rho(t)^{-1}y = 0$$

is disconjugate on $(0,\infty)$ ([14] Section XI.6). Plug in the expressions for ρ and α_1 to find that this Euler-Lagrange equation reads

$$t^2y'' + \frac{4}{3}ty' + \frac{2}{9}\nu y = 0 \tag{25}$$

which has solutions t^r where r a root of the indicial equation

$$r^2 + \frac{1}{3}r + \frac{2}{9}\nu = 0.$$

Equation (25) fails to be disconjugate if and only if r has an imaginary part, which is to say iff and only if the discriminant, $\Delta = (1-8\nu)/9$ is negative. Δ is negative if and only if $\nu > \frac{1}{8}$ in which case the solutions of (25) are

$$y(t) = At^{-1/6}\cos(a\ln t) + Bt^{-1/6}\sin(a\ln t); \quad a^2 = \frac{1}{4}|\Delta| = \frac{1}{36}(8\nu - 1).$$

which has in fact infinitely many conjugate points on $(0,\infty)$.

To finish the instability part of the proof of lemma 2(A) just note that our instability condition $\nu > \frac{1}{8}$ is precisely the condition for spiraling at the Eulerian restpoint.

It remains to check variational stability for $\nu \le \frac{1}{8}$. Consider an arbitrary variation of the Euler homothetic motion. Factoring out $\rho(t)$ it can be written in the form (22) for some $v(t)$. In the planar case, we can expand $v(t)$ as

$$\sum_{i=1}^{6} \varphi_i(t) e_i$$

where $e_1, \ldots, e_6$ is a basis for the configuration space $\mathbb{R}^6$ consisting of eigenvectors for $D\tilde{\nabla} U(c)$, orthogonal with respect to the mass metric. Such a basis exists since the linear map $D\tilde{\nabla} U(c)$ is symmetric with respect to the mass metric. We can take $e_3 = c$ with eigenvalue $\alpha_3 = 2U(c)$, $e_4 = c^{\perp}$ with eigenvalue $\alpha_4 = 0$ and e_5, e_6 independent translation vectors with eigenvalues $\alpha = U(s)$. The other two eigenvalues are $\alpha_1 = -\nu U(c), \alpha_2 = (3+2v)U(c)$. The second variation decomposes as a sum

$$\frac{d^2 A(\gamma^\epsilon; a, b)}{d\epsilon^2}\Big|_{\epsilon=0} = \sum_{i=1}^{6} Q_i(\varphi_i; a, b) = \sum_{i=1}^{6} \int_a^b \left[\rho(t)^2 \dot{\varphi}_i(t)^2 + \alpha_i \rho(t)^{-1} \varphi_i(t)^2\right] dt.$$

Now α_1 is the smallest of the four eigenvalues and it follows that for any $\varphi(t)$, $Q_i(\varphi; a, b) \ge Q_1(\varphi; a, b)$. So it suffices to show that $Q_1(\varphi; a, b)$ is positive definite. This was shown in the first part of the proof. For variations in $\mathbb{R}^d$, the same eigenvalues occur but with different multiplicities, so the same argument works.

□

Remark on Hessians. Some words are in order regarding the term $D\tilde{\nabla} U$ occurring in the formula and its relation to the Hessian $D^2 U$. If f is any smooth function on a real vector space $\mathbb{E}$ then its Hessian $D^2 f(p)$ at $p \in \mathbb{E}$ is the coordinate independent bilinear symmetric form defined by

$$\frac{d^2}{d\epsilon^2} f(p + \epsilon v)|_{\epsilon=0} = (D^2 f(p))(v, v).$$

If $\langle \cdot, \cdot \rangle$ is any inner product on $\mathbb{E}$, then $D^2 f(p)(v, v) = \langle D\nabla f(p)(v), v\rangle$ where ∇f is the gradient vector field of f with respect to the inner product, and $D\nabla f$ is the derivative (Jacobian) of the vector field $X = \nabla f$ on the vector space $\mathbb{E}$ as given by

$$DX(p)(v) = \frac{d}{d\epsilon}|_{\epsilon=0} X(p + \epsilon v).$$

4.2. **Proof of Part (B) of Lemma 2.** Part (B) of Lemma 2 follows from Proposition 4.3 in [1], however that paper used an estimate taken from the literature. Our proof is just an extension of that of part (A) using an estimate explained with what we had in the previous section.

Proof. We now consider a parabolic motion asymptotic to an Euler central configuration c. Shifting the origin of time, if necessary, such a solution can be written as $\alpha_0(t) = \rho(t)(c + \beta(t))$, $t \ge t_0 > 0$, where $\rho(t)c$ is the parabolic homothetic solution and $\beta(t) = O(t^{-d})$ as $t \to \infty$ for some $d > 0$.

A bit of explanation is in order regarding β's rate of convergence to zero. If we represent α_0 in McGehee (spherical) coordinates we get $\alpha_0(t) = r(t)s(t)$ with $s(t)$ normalized, $s(t) \to c$ as $t \to \infty$. We are also supposed to parameterize the curve using τ instead of t. The stable manifold theorem applied to the equilibrium point for c yields $|s(\tau) - c| \le Ae^{-\mu\tau}$ for τ sufficiently large, where $\mu > 0$ is any number such that $-\mu$ is greater than all of the negative eigenvalues from proposition 2 applied to c. Our two representations of α_0 are related by

$$r(t) = \rho(t)\sqrt{1 + |\beta(t)|^2}, \quad s(t) = (c + \beta(t))/\sqrt{1 + |\beta(t)|^2}.$$

Integrating the relation $r^{3/2}d\tau = dt$ we have that $t = Ae^{3v\tau/2} + \eta$ where $v = \sqrt{2U(c)}$ and where η is exponentially small relative to the first term. Since $e^{-\mu\tau} \sim t^{-\frac{2}{3}\frac{\mu}{v}}$ it follows that $\beta(t) \to 0$ at a rate t^{-d} with $d = 2\mu/3v$. Now take a variation

$$\alpha_\epsilon(t) = \rho(t)(c + \beta(t) + \epsilon v(t)) \text{ with } v(t) = \varphi(t)z$$

as before. So

$$\frac{d\alpha_\epsilon(t)}{d\epsilon} = \rho(t)v(t),\ \frac{d^2\alpha_\epsilon(t)}{d\epsilon^2}\Big|_{\epsilon=0} = 0$$

$$\begin{aligned}\frac{d^2A(\alpha_\epsilon; a, b)}{d\epsilon^2}\Big|_{\epsilon=0} &= \int_a^b \rho(t)^2\langle\dot v(t), \dot v(t)\rangle_m, dt \\ &+ \int_a^b \rho(t)^{-1}[U(c)\langle v(t), v(t)\rangle_m + D^2U(c+\beta(t))(v(t), v(t))]\, dt \\ &= Q(\varphi; a, b) + \int_a^b (D^2U(c+\beta(t)) - D^2U(c))(z, z)\frac{\varphi(t)^2}{\rho(t)}\, dt\end{aligned}$$

with Q as per equation (24). (We have written the Hessian of U in the form D^2U as per the remark above on Hessians, rather than in the $D\nabla_m U$ form.) According to Part (A) there are $[a, b] \subset (0, \infty)$, $\varphi_1 \in C^2([a, b])$ such that $\varphi_1(a) = \varphi_1(b) = 0$ and $Q(\varphi_1; a, b) < 0$.

Defining $\varphi_\lambda(t) = \varphi_1(\frac{t}{\lambda})$ we have $\dot\varphi_\lambda(t) = \lambda^{-2}\dot\varphi_1(\frac{t}{\lambda})$

$$Q(\varphi_\lambda; \lambda a, \lambda b) = \int_{\lambda a}^{\lambda b} \lambda^{-\frac{2}{3}}\left[\rho(\tfrac{t}{\lambda})^2\dot\varphi_1(\tfrac{t}{\lambda})^2 - \mu\rho(\tfrac{t}{\lambda})^{-1}\varphi_1(\tfrac{t}{\lambda})^2\right]dt = \lambda^{\frac{1}{3}}Q(\varphi_1; a, b)$$

For λ sufficiently large and $t \geq \lambda a$ we have that $|\beta(t)| \leq C_1t^{-d}$ and so

$$\|D^2U(c+\beta(t)) - D^2U(c)\| \leq C_2t^{-d}.$$

Thus

$$\int_{\lambda a}^{\lambda b} \|D^2U(c+\beta(t)) - D^2U(c)\|\frac{\varphi_\lambda(t)^2}{\rho(t)}dt \leq C_2\int_{\lambda a}^{\lambda b}\frac{\varphi_\lambda(t)^2}{\rho(t)t^d}dt = C_2\lambda^{\frac{1}{3}-d}\int_a^b\frac{\varphi_1(s)^2}{\rho(s)s^d}ds.$$

Using $v(t) = \varphi_\lambda z$ we have

$$\frac{d^2A(\alpha_\epsilon; \lambda a, \lambda b)}{d\epsilon^2}\Big|_{\epsilon=0} \leq \lambda^{\frac{1}{3}}\left(Q(\varphi_1; a, b) + C_2\lambda^{-d}\int_a^b\frac{\varphi_1(s)^2}{\rho(s)s^d}ds\right) < 0.$$

for λ sufficiently large. □

5. Symplectic Structure, Lagrangian submanifolds, and Proofs of lemma 1 and theorem 1.

The differential equations of the planar three-body problem preserve the standard symplectic structure on $\mathbb{R}^{6d}$

$$\omega = m_1dq_1 \wedge dv_1 + m_2dq_2 \wedge dv_2 + m_3dq_3 \wedge dv_3.$$

Here, as usual, the wedge of vectors of one forms means adding the componentwise wedges so, for example, $(dx, dy) \wedge (du, dv) = dx \wedge du + dy \wedge dv$. The restriction of the flow to $X = \mathbb{E} \times \mathbb{E}$ preserves the restriction of ω. The pullback of ω under the change of variables $q_i = r\, s_i, v_i = r^{-\frac{1}{2}}z_i$ is

$$\Omega_r = \sum_i m_i\left(r^{\frac{1}{2}}\, ds_i \wedge dz_i + r^{-\frac{1}{2}}dr \wedge s_i \cdot dz_i + \frac{1}{2}r^{-\frac{1}{2}}dr \wedge z_i \cdot ds_i\right)$$

where, for example, $s_i \cdot dz_i = (s_{i1}, s_{i2}) \cdot (dz_{i1}, dz_{i2}) = s_{i1}dz_{i1} + s_{i2}dz_{i2}$.

If we use $u = 1/r$ instead we get

$$\Omega_u = \sum_i m_i \left(u^{-\frac{1}{2}}\, ds_i \wedge dz_i + u^{-\frac{3}{2}} s_i \cdot dz_i \wedge du + \frac{1}{2} u^{-\frac{3}{2}} z_i \cdot ds_i \wedge du \right). \tag{26}$$

In both cases we restrict to the $4d$ dimensional subset X of $\mathbb{R}^{6d+1}$ where $r > 0, u > 0$ and where the normalizations (14) hold.

Lemma 4. *Let $p(\tau)$ be any solution of the blown-up differential equations with $r(\tau) > 0, u(\tau) > 0$ and let vectorfields $a(\tau), b(\tau)$ be solutions of the variational equations along $p(\tau)$ which are tangent to an energy manifold. Then $\Omega_r(p(\tau))(a(\tau), b(\tau))$ and $\Omega_u(p(\tau))(a(\tau), b(\tau))$ are constant.*

Proof. Let ξ denote the vectorfield on $\mathbb{R}^{6d+1}$ given by (11), (12), (13). Let $\eta = r^{-\frac{2}{3}}\xi$ be the same vectorfield without the change of timescale. Since η is the pullback of the Hamiltonian field, it preserves the pullback form Ω_r. In other words, the Lie derivative

$$L_\eta \Omega_r = \frac{d}{dt}\phi_t^* \Omega_r|_{t=0} = 0.$$

Since $\xi = f\eta$, where $f = r^{\frac{3}{2}}$, Cartan's formula gives

$$L_\xi \Omega_r = d(\iota_\xi \Omega_r) + \iota_\xi d\Omega_r = d(f\, \iota_\eta \Omega_r) + 0 = d(f dH) = df \wedge dH.$$

Here we used the fact that $\iota_\eta \Omega_r = dH$ which is the pullback of the differential form definition of Hamilton field.

If p, a, b are as in the statement of the lemma then

$$\frac{d}{d\tau}\Omega_r(p)(a,b) = L_\xi \Omega_r(p)(a,b) = (df \wedge dH)(p)(a,b) = 0$$

since $dH(p)(a) = dH(p)(b) = 0$. □

Proposition 6. *Let $l = (u, s, z) = (0, s, vs)$ be one of the Lagrange restpoints at infinity with $v > 0$ and let $W^s_+(l)$ denote the part of the stable manifold with $u > 0$. Then $W^s_+(l)$ is a $2d$-dimensional invariant manifold which is a Lagrangian submanifold of X. Similarly, at the restpoints with $v < 0$ the unstable manifold $W^u_+(l)$ is Lagrangian.*

Proof. For the planar problem, $d = 2$, Propositions 2 and 3 give the eight eigenvalues of the variational equations at l. For positive masses, the quantity k from proposition 3 satisfies $0 \le k < 1$. It follows that if $v > 0$ then there are three positive eigenvalues

$$v, \frac{-v}{4}\left(1 - \sqrt{13 \pm 12\sqrt{k}}\right)$$

four negative eigenvalues

$$-v, -\frac{v}{2}, \frac{-v}{4}\left(1 + \sqrt{13 \pm 12\sqrt{k}}\right)$$

and one zero eigenvalue. The latter is due to the rotational symmetry. In fact l is part of a circle of equilibria. This circle is normally hyperbolic ([15], p.1) so each equilibrium has a four-dimensional stable manifold. The discussion at the end of section 3.3 shows that for $d > 2$ we have normally hyperbolic manifolds $(T_1\mathbb{S}^{d-1})$ of equilibria and that each equilibrium point has a stable manifold of dimension $2d$ in the $4d$-dimensional phase space. The eigenvalues are the same as for the planar case, but with different multiplicities.

The first negative eigenvalue $-v$ has eigenvector $(\delta u, \delta s, \delta z) = (1, 0, 0)$ and it follows that the stable manifold has an open subset $W^s_+(l)$ with $u > 0$. Moreover, the stable eigenvectors are in the subspace $\delta H = 0$. It follows that $W^s_+(l)$ is contained in the energy manifold $\{H = 0\}$. Using blown-up coordinates we need to show that the two-form Ω_u vanishes on tangent vectors to $W^s_+(l)$. Let

a_0, b_0 be two tangent vectors to $W^s_+(l)$ at a point $p_0 \in W^s_+(l)$. To show that $\Omega_u(p_0)(a_0, b_0) = 0$ it suffices, by lemma 4, to show that $\Omega_u(p(\tau))(a(\tau), b(\tau)) \to 0$ as $\tau \to \infty$. For this we need estimates on the exponential decay of u and the components of a, b. We will denote the components of $a(\tau)$ and $b(\tau)$ by $\delta s_i(\tau)$ and $\delta z_i(\tau)$ respectively.

The four stable eigenvalues of l are $-v, -v/2$ and $\lambda_\pm = \frac{-v}{4}\left(1 + \sqrt{13 \pm 12\sqrt{k}}\right)$. The largest of these is $-v/2$. For the planar case, this eigenvalue is associated to the eigenvector $(\delta s, \delta z) = (s^\perp, ks^\perp)$. For $d > 2$ there are $2d - 3$ such eigenvectors, spanning the tangent space to the action of $\mathbf{SO}(d)$. Let $\mu = \max(-v, \lambda_+, \lambda_-) < -v/2$ be the largest of the remaining eigenvalues. We will show that there are constants α, β such that

$$0 > -v/4 > \alpha > -v/2 > \beta > \mu$$

and positive constants c_i such that

$$\begin{aligned} u^{-\frac{1}{2}} &\leq c_1 \exp(\frac{1}{2}v\tau) \qquad u^{-\frac{3}{2}} \leq c_1 \exp(\frac{3}{2}v\tau) \qquad |\delta u| \leq c_2 \exp(-v\tau) \\ |\delta s_i(\tau)| &\leq c_3 \exp(\alpha\tau) \qquad |\delta z_i(\tau)| \leq c_3 \exp(\alpha\tau) \\ |z_i \cdot \delta s_i(\tau)| &\leq c_4 \exp(\beta\tau) \qquad |s_i \cdot \delta z_i(\tau)| \leq c_4 \exp(\beta\tau). \end{aligned} \tag{27}$$

Substituting these estimates into the formula (26) for $\Omega_u(p(\tau))(a(\tau), b(\tau))$, we find that all three terms tend to 0 as $\tau \to \infty$, as required.

It remains to prove the estimates (27). Since $u' = -vu$ and $v(\tau)$ converges exponentially to the value v at the restpoint we have bounds $c\exp(-v\tau) \leq u(\tau) \leq c_2 \exp(-v\tau)$ for some constant $c > 0, c_2 > 0$ which depend on the particular solution $p(\tau)$ under consideration. To see this note that

$$u(\tau)\exp(v\tau) = u(0)\exp(\int_0^\tau (v - v(s))\, ds)$$

and the integral is bounded above and below since the integrand tends to 0 exponentially. Since $u(0) > 0$ we get a positive lower bound c. The lower bound on u gives the upper bounds on $u^{-\frac{1}{2}}$ and $u^{-\frac{3}{2}}$ in (27). Note that the variational equation for $\delta u(\tau)$ is also $\delta u' = -v\delta u$ so the same reasoning gives the required upper bound on $|\delta u|$.

Next consider the flow inside the stable manifold $W^s_+(l)$. Choose a basis for the tangent space of $W^s_+(l)$ at l consisting of stable eigenvectors $e_1, \ldots, e_{2d}$, so that matrix of the linearized equations is block-diagonal: $\mathrm{diag}(-v, -v/2, \lambda_+, \lambda_-)$. Let $w = (w_1, \ldots, w_{2d})$ be coordinates with respect to this basis and extend these to local coordinates on $W^s_+(l)$ (which is a smooth graph over the tangent space). The differential equations are now of the form $w_i' = \lambda_i w_i + \ldots$ where the omitted terms are of order at least 2. Now since $-v/2$ is the largest eigenvalue, it is a standard argument to show that for any $\alpha > -v/2$ we will have estimates

$$|w(\tau)| \leq c\exp(\alpha\tau) \qquad |\delta w(\tau)| \leq c\exp(\alpha\tau)$$

in some neighborhood of l. It is no problem to choose $\alpha < -v/4$. Since $\delta s, \delta z$ are smooth functions of the δw_i, we get the estimates on the second line of (27). We also get exponential convergence $|p(\tau) - l| = O(\exp(\alpha\tau))$. It follows that the higher order terms in w_i which will be dropped in the differential equations below will be of order $O(\exp(2\alpha\tau))$.

The estimates on the last line are a bit trickier. There is a linear projection P_j which reads off the δz_j components of vectors. Then for any solution of the variational equations along the stable manifold, we will have

$$\delta z_j(\tau) = \sum_{i=1}^{2d} \delta w_i(\tau) P_j e_i + O(\exp(2\alpha\tau)).$$

Recall that the coordinates of l are $(u,s,z)=(0,s_0,vs_0)$ where $s_0=(s_{01},s_{02},s_{03})$ is the Lagrange configuration and that the eigenvectors e_i of the weak eigenvalue $-v/2$ are tangent to the action of the rotation group. Hence $s_{i0}\cdot Pe_i=0$ for these eigenvalues. Since $s_i(\tau)=s_{i0}+O(\exp(\alpha\tau))$ we have

$$s_j(\tau)\cdot\delta z_j(\tau)={\sum}'\delta w_i(\tau)P_je_i+O(\exp(2\alpha\tau))$$

where the sum is over the indices associated to the eigenvalues other than $-v/2$. A similar argument applies to $z_j(\tau)\cdot\delta s_j(\tau)$.

To complete the proof we will show that the remaining components δw_i decay at the faster rate $O(\exp(\beta\tau))$. The differential equations for these functions are of the form $\delta w_i'=\lambda_i\delta w_i+O(\exp(2\alpha\tau))$ with $\lambda_i\le\mu<\beta$. Since $2\alpha<-v/2$ it is no problem to choose β which also satisfies $2\alpha<\beta<-v/2$. Then it is a standard estimate to get $|\delta w_i(\tau)|=O(\exp(\beta\tau))$.

□

Proof of Lemma 1. From lemma 3 we see that the stable space of a Lagrange restpoint at infinity is generated by the eigenvector $(1,0,0)$ and $2d-1$ eigenvectors $(0,\delta s_\alpha,k_{\alpha-}\delta s_\alpha)$ for eigenvectors δs_α of the eigenvalues α of $D\tilde{\nabla}U(s)$. Thus the projection of the stable space is the whole tangent space at $(0,c)$ of the configuration space. It follows from the implicit function theorem that $W^s_+(l)$ is a graph near infinity. More precisely, there is a product neighborhood $\bar{V}$ of $(0,c)$ in the blown-up configuration space $[0,\infty)\times S^{2d-1}$ and a smooth map $(u,s)\mapsto y(u,s)$ from $\bar{V}$ to the space $\mathbb{E}$ of blown up velocities such that the graph of this map coincides with the stable manifold of l in some neighborhood of l. Now being Lagrangian does not make sense at $u=0$ since the symplectic structure explodes, so in the statement of lemma 2, when we say that $W^s_{loc}(l)$ is a "Lagrangian graph" we mean over $V=\bar{V}\setminus\{u=0\}$. We may choose V such that there is a differentiable function f defined on V such that

$$W^s_{loc}(l)\cap(V\times\mathbb{E})=\operatorname{graph}df \tag{28}$$

and this set is positively invariant. □

Definition 6. *By a "neighborhood of c at infinity" we mean a neighborhood of the form described in the end of the proof immediately above. When expressed in $\mathbb{E}$ such a neighborhood is a truncated open cone consisting of those points $q\in\mathbb{E}$ of the form $q=rs$ where $|s|=1$, $|s-c|<\delta$ and $u=1/r<1/R$.*

Proof of Corollary 1. $W^s_{loc}(l)$ is an exact Lagrangian graph over some neighborhood $V\subset\mathbb{E}$ of infinity, as per the above terminology.

For any $x\in V$ there is a unique motion $\gamma(t)$ with $\gamma(0)=x$, $(x,\dot\gamma(0)^*)\in W^s_{loc}(l)$ and it is given by the solution to $\dot\gamma^*=df(\gamma)$ with $\gamma(0)=x$. Here w^* denotes the dual of w with respect to the mass metric – which is to say - the inverse Legendre transform of w relative to our Lagrangian.

Claim 1. *There is a neighborhood $V_1=\{ru: r>R_1, |u-c|<\delta_1\}$ such that if $x\in V_1$ and $\alpha:[0,\infty)\to\mathbb{E}$ is the curve calibrated by B_c with $\alpha(0)=x$, we have that $(\alpha,\dot\alpha^*)$ lies on $W^s_{loc}(l)\cap(V_1\times\mathbb{E})$*

Let $U(c)=U_c$, and consider the homotetic motion $\gamma_c(t)=t^{\frac{2}{3}}(\frac{9}{2}U_c)^{\frac{1}{3}}c$.

Consider the one-dimensional Kepler's problem with potential energy U_c/r, its corresponding finite time action h_l and action potential $S_1(r,s)=\sqrt{8U_c}(\sqrt{s}-\sqrt{r})$. We have

$$S_1(\|x\|,\|y\|)\le h_1(\|x\|,\|y\|;\tau)\le h(x,y;\tau)$$

Given $\epsilon>0$ such that $3\sqrt{U_c}>\sqrt{U_c+\epsilon}(2+\epsilon)$, we can choose $0<\delta_1<\epsilon$ such that $U(u)<U_c+\epsilon$ for $|u-c|<\delta_1$. Choose $R_1>(3\sqrt{U_c}-\sqrt{U_c+\epsilon}(2+\delta))^{-2}U_c4R$ such that $W^s_{loc}(l)\cap(V_1\times\mathbb{E})$ is positively invariant. where $V_1=\{ru: r>R_1, |s-c|<\delta_1\}$.

Consider the Kepler's problem on $(\mathbb{R}^d)^N$ with Lagrangian

$$L_c(x,v) = \frac{\|v\|^2}{2} + \frac{U_c+\epsilon}{\|x\|}$$

Let $x = ru \in V_1$ and identify the plane that contains γ_c and x with $\mathbf{C}$, identifying γ_c with $\mathbb{R}^+$, then write $u = e^{i\theta}$ with $|\theta| < \delta_1$.

If $\gamma : [0,b] \to V_1$ connects x to sc, then $A_L(\gamma) \le A_{L_c}(\gamma)$.

We can choose γ in the plane C such that

$$\begin{aligned} A_{L_c}(\gamma) &= \sqrt{8(U_c+\epsilon)(r+s-2\sqrt{rs}\cos(\frac{\theta}{2}))} = \sqrt{8(U_c+\epsilon)((\sqrt{s}-\sqrt{r}\cos(\frac{\theta}{2}))^2 + r\sin^2(\frac{\theta}{2}))} \\ &\le \sqrt{8(U_c+\epsilon)}((\sqrt{s}-\sqrt{r}\cos(\frac{\theta}{2})) + \sqrt{r}|\sin(\frac{\theta}{2})| < \sqrt{8(U_c+\epsilon)}((\sqrt{s}-\sqrt{r}) + \sqrt{r}\delta_1) \end{aligned}$$

Let $\omega : [0,a] \to \mathbb{E}$ be a curve that conects x and sc with $\|\omega(\tau)\| \le R$ for some τ. Then

$$A_L(\omega) \ge h(x,\omega(\tau);\tau) + h(\omega(\tau), sc; a-\tau) \ge \sqrt{8U_c}(\sqrt{r}-\sqrt{R}+\sqrt{s}-\sqrt{R})$$

Consider first the case $s = 4r$

$$\begin{aligned} A_{L_c}(\gamma) &\le \sqrt{8(U_c+\epsilon)r}(2+\delta_1)) \\ A_L(\omega) &\ge \sqrt{8U_c}(3\sqrt{r}-2\sqrt{R}) \end{aligned}$$

For $r > R_1$

$$A_L(\gamma) \le A_{L_c}(\gamma) < A_L(\omega).$$

In the case $s > 4r$ we use for $hat\gamma$ a curve γ that connects x to $4rc$ as above, followed by the segment of γ_c from $4rc$ to sc. Then

$$A_L(\hat{\gamma}) \le A_L(\gamma) + \sqrt{8U_c}(\sqrt{s}-2\sqrt{r})$$

so, for $r > R_1$, in any case

$$A_L(\hat{\gamma}) < A_L(\omega).$$

Thus, for $r > R_1$ the curves y_T with $A_L(y_T) = \phi(x, \gamma_c(T))$ defined in [31] never leave V, and then the limit curve that calibrates the Busemann function never leaves V.

We know by Propositions 1, 5 that the solution $(\alpha, \dot{\alpha}^*)$ lies on the stable manifold $W^s_+(l)$, so eventually enters $W^s_{loc}(l)$. If we had $(x, \dot{\alpha}^*(0)) \in W^s_+(l) \setminus W^s_{loc}(l)$, then α would have to leave V before $(\alpha, \dot{\alpha}^*)$ enters $W^s_{loc}(l)$, which we have seen is not possible, and the claim is proved.

For $t > 0$, B_c is differentiable at $\alpha(t)$ and $dB_c(\alpha(t)) = \dot{\alpha}(t)^*$. Since $(\alpha(t), \dot{\alpha}(t)^*) \in W^s_{loc}(l)$, (28) gives $\dot{\alpha}(t)^* = df(\alpha(t))$. Thus $dB_c(\alpha(t)) = df(\alpha(t))$, proving the corollary. (We also have that $B_c = f + k$ on V, k a constant.) □

Proof of Theorem 1. Let $\gamma(t)$ tend parabolically to a Lagrange central configuration c and let V be the neighborhood of c at infinity as in Corollary 1 and lemma 2. Then, since $(\gamma(t), \dot{\gamma}(t)^*)$ lies on the stable manifold $W = W^s_+(l)$ we must have that $\gamma([T,\infty)) \subset V$ for T large enough. Consequently for $t \ge T$ we have that $\dot{\gamma}(t)^* = dB_c(\gamma(t))$. The curve $\alpha : [0,\infty) \to \mathbb{E}$ calibrated by B_c that starts at $\gamma(T)$ is also a solution of the differential equation $\dot{z}^* = dB_c(z)$. By uniqueness of solutions we have $\alpha(t) = \gamma(t+T)$. Then $\gamma : [T,\infty) \to \mathbb{E}$ is calibrated by B_c and in particular it is a free time minimizer. □

6. Appendix

The appendix provides the proofs of the propositions about eigenvalues used above.

Proof of Proposition 2. The first two eigenvalues in the list are from $(\delta r, \delta s, \delta z) = (1, 0, 0)$ and $(0, 0, s)$. The others come from the eigenvalues of B found in the lemma. The eigenvalues $-v, 0$ come from the eigenvector $\delta s = s^\perp$ with $\alpha = 0$ and the other two come from the two nontrivial eigenvalues.

For the restpoints at infinity, the computation is the same except that the first eigenvalue on the list is now associated to $(\delta u, \delta s, \delta z) = (1, 0, 0)$ and has eigenvalue $-v$ instead of v. □

To prove propositions 3 and 4 we need to find the nontrivial eigenvalues α_1, α_2 of $D\tilde{\nabla} U(s)$ for the equilateral and collinear central configurations of the three-body problem. These can be deduced from the work of Siegel but we will give a quick discussion here.

It is straightforward to calculate the 6×6 matrix $D\nabla U(s)$ with the result

$$D\nabla U(s) = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{23} \end{bmatrix} \tag{29}$$

where the 2×2 blocks are

$$D_{ij} = \frac{m_i m_j}{r_{ij}^3}\left(I - 3u_{ij}u_{ij}^t\right), \quad u_{ij} = \frac{s_i - s_j}{r_{ij}} \qquad \text{for } i \neq j$$

and

$$D_{ii} = -\sum_{j \neq i} D_{ij}.$$

It is more convenient to work with the matrix

$$P = \frac{I(s)}{U(s)} M^{-1} D\nabla U(s).$$

Since P is invariant under scaling and translation, it can be computed without imposing the normalizations (14). If β is an eigenvalue of P then $\alpha = U(s)(\beta + 1)$ is an eigenvalue of $D\tilde{\nabla} U(s)$ for the corresponding normalized s. So we are reduced to finding the nontrivial eigenvalues β_1, β_2 of P.

Proof of Proposition 3. Consider an equilateral triangle configuration s. Working with P we can use the unnormalized configuration

$$s_1 = (1, 0) \qquad s_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad s_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

for which

$$U(s) = \frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{\sqrt{3}} \qquad I(s) = \frac{3(m_1 m_2 + m_1 m_3 + m_2 m_3)}{m}.$$

Using these together with (29) gives

$$P = \frac{1}{4m}\begin{bmatrix} 5(m_2+m_3) & 3\sqrt{3}(m_3 - m_2) & -5m_2 & 3\sqrt{3}m_2 & -5m_3 & -3\sqrt{3}m_3 \\ 3\sqrt{3}(m_3 - m_2) & -(m_2+m_3) & 3\sqrt{3}m_2 & m_2 & -3\sqrt{3}m_3 & m_3 \\ -5m_1 & 3\sqrt{3}m_1 & 5m_1 - 4m_3 & -3\sqrt{3}m_1 & 4m_3 & 0 \\ 3\sqrt{3}m_1 & m_1 & -3\sqrt{3}m_1 & -m_1 + 8m_3 & 0 & -8m_3 \\ -5m_1 & -3\sqrt{3}m_1 & 4m_2 & 0 & 5m_1 - 4m_2 & 3\sqrt{3}m_1 \\ -3\sqrt{3}m_1 & m_1 & 0 & -8m_2 & 3\sqrt{3}m_1 & -m_1 + 8m_2 \end{bmatrix}$$

One can guess 4 of the 6 eigenvalues of P. If $e_1 = (1,0), e_2 = (0,1)$ then (e_1, e_1, e_1) and (e_2, e_2, e_2) are eigenvectors with eigenvalue $\beta = 0$. Also $s, s^\perp$ are eigenvectors with eigenvalues $\beta = 2, -1$ respectively. Since the trace of P is 2, the remaining eigenvalues satisfy $\beta_1 + \beta_2 = 1$. Alternatively, the numbers $\gamma_i = \beta_i + 1$ which we really want, satisfy $\gamma_1 + \gamma_2 = 3$. We can also find the product $\gamma_1\gamma_2$ as follows. We have

$$(\operatorname{trace} P)^2 - \operatorname{trace} P^2 = (1 + \beta_1 + \beta_2)^2 - (5 + \beta_1^2 + \beta_2^2) = 2\beta_1\beta_2 - 2 = 2\gamma_1\gamma_2 - 6.$$

With some computer assistance, this gives

$$\gamma_1\gamma_2 = \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{4(m_1 + m_2 + m_3)^2}$$

Solving the quadratic equation $\gamma^2 - 3\gamma + \gamma_1\gamma_2 = 0$ gives the eigenvalues $\alpha_i = U(s)\gamma_i = v^2\gamma_i/2$ of $D\tilde{\nabla}U(s)$ listed in the proposition. Then we get the nontrivial eigenvalues λ of the equilibrium from proposition 2. □

Proof of proposition 4. Consider a normalized collinear central configuration such that $s_i = (x_i, 0) \in \mathbb{R}^2$. Then the unit vectors $u_{ij} = (\pm 1, 0)$ so the 2×2 matrices D_{ij} reduce to

$$D_{ij} = \frac{m_i m_j}{r_{ij}^3} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \qquad i \neq j.$$

Rearranging the variables as $q = (x_1, x_2, x_3, y_1, y_2, y_3)$ produces a block structure

$$M^{-1} D\nabla U(s) = \begin{bmatrix} 2C & 0 \\ 0 & -C \end{bmatrix} \tag{30}$$

where C is the $n \times n$ matrix

$$C = \begin{bmatrix} \frac{m_2}{r_{12}^3} + \frac{m_3}{r_{13}^3} & -\frac{m_2}{r_{12}^3} & -\frac{m_3}{r_{13}^3} \\ -\frac{m_1}{r_{12}^3} & \frac{m_1}{r_{12}^3} + \frac{m_3}{r_{23}^3} & -\frac{m_3}{r_{23}^3} \\ -\frac{m_1}{r_{13}^3} & -\frac{m_2}{r_{23}^3} & \frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3} \end{bmatrix}.$$

An eigenvalue μ of C determines two eigenvalues

$$\alpha = -\mu + U(s), 2\mu + U(s)$$

for $D\tilde{\nabla}U(s) = M^{-1}D\nabla U(s) + U(s)I$.

It is possible to guess two eigenvectors of C. First of all $v_1 = (1,1,1)$ is an eigenvector with eigenvalue 0. Next, let $v_2 = (x_1, x_2, x_3)$ be the vector of x-coordinates of the collinear central configuration. Then it is easy to see that

$$Cv_2 = -{M_0}^{-1}\nabla_x U$$

where ∇_x is the partial gradient with respect to the x-coordinates and $M_0 = \operatorname{diag}(m_1, m_2, m_3)$. Since s is a normalized central configuration, we have $Cv_2 = U(s)v_2$, so v_2 is also an eigenvector, with eigenvalue $U(s)$. The remaining, nontrivial eigenvalue of C can now be found as $\mu = \tau - U(s)$ where $\tau = \operatorname{trace}(C)$, i.e.,

$$\tau = \left(\frac{m_1 + m_2}{r_{12}^3} + \frac{m_1 + m_3}{r_{13}^3} + \frac{m_2 + m_3}{r_{23}^3} \right).$$

Therefore the nontrivial eigenvalues of of $D\tilde{\nabla}U(s)$ are

$$\alpha = 2\tau - U(s), 2U(s) - \tau.$$

To get the form shown in the proposition, let ν be the translation and scale invariant quantity

$$\nu = \frac{I(s)}{U(s)}\tau - 2.$$

Then for the normalized configuration $\alpha_1, \alpha_2 = -U(s)\nu, U(s)(3+2\nu)$ and it remains to show that ν has the indicated form.

We just indicate a computer assisted way to prove it. Using the configuration $s_1 = (0,0), s_2 = (r,0), s_3 = (1+r,0)$ we have

$$r_{12} = r \qquad r_{23} = 1 \qquad r_{13} = 1+r.$$

Substituting these into the formulas for $I(s), U(s), \tau$ expresses $\nu = \frac{I(s)}{U(s)}\tau - 2$ as a rational function $\nu(r)$. Subtracting the expression (3) and factorizing the difference reveals that there is a factor of $g(r)$ in the numerator, where $g(r)$ is the fifth degree polynomial (2) giving the location of the central configuration. So $\nu(r)$ is indeed given by (3) at the central configuration. □

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