SHOOTING FOR THE EIGHT – A TOPOLOGICAL EXISTENCE PROOF FOR A FIGURE-EIGHT ORBIT OF THE THREE-BODY PROBLEM

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Abstract. A topological existence proof is given for a figure-eight periodic solution of the equal mass three-body problem. The proof is based on the construction of a Wazewski set W in the phase space. The figure-eight solution is then found by a kind of shooting argument in which symmetrical initial conditions entering W are followed under the flow until they exit W. A linking argument shows that the image of the symmetrical entrance states under this flow map must intersect an appropriate set of symmetrical exit states.

1. Introduction

The goal of this paper is to give a topological proof of existence of a figure-eight type periodic solution of the equal-mass three-body problem. A variational existence proof for such an orbit was given by Chenciner and Montgomery [5]. The orbit is such that the three masses chase one another around a single figure-eight shaped curve (see figure 1). It has zero angular momentum and, by adjusting the size and speed, may be given an arbitrary negative energy.

In Chenciner and Montgomery's proof, a 12-fold symmetry plays an important role and the same is true here. The orbit can be parametrized in such a way that it begins at the Eulerian central configuration with one mass at the midpoint of the other two, for example with mass m_3 between m_1, m_2. During the first twelfth of the period the masses move to an isosceles configuration with m_1 on the axis of symmetry (see figure 1). Now the rest of the orbit can be obtained by reflection, translation and permutation of the masses. For example, during the next twelfth of the period, the masses return to an Eulerian configuration, but this time with m_2 in the middle. The motion during the second twelfth of the period is obtained from the motion in the first twelfth by permuting m_2 and m_3, reversing time, and reflecting in the long axis of the eight.

The method of proof used here is a variation of an idea used by Conley in the restricted three-body problem [2]. In Conley's paper, the retrograde lunar orbit of Hill is shown to exist for a wide range of values of the Jacobi constant. After regularizing double collisions, the problem becomes one of finding a solution of a system of second-order differential equations in the plane which moves from the positive x-axis to the positive y-axis across the first quadrant and meets both axes orthogonally.

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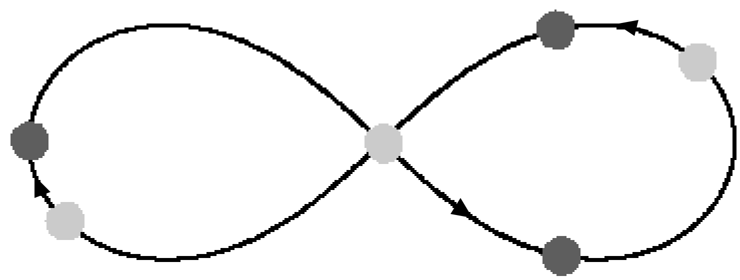

Figure 1. Figure eight orbit. In the first twelfth of the period, the configuration changes from collinear (light gray dots) to isosceles (dark gray dots).

Such a solution is found by a shooting argument. Points starting orthogonal to the x-axis are followed until one of two exit conditions holds – they either hit the positive y-axis or their velocity vectors become horizontal. As the initial point along the x-axis varies, the final behavior changes from hitting the y-axis with nonzero slope to having a horizontal velocity vector before hitting the positive y-axis. Somewhere in between, there must be a point whose velocity becomes horizontal exactly when it reaches the positive y-axis and this gives the desired periodic solution.

The main difficulty in this approach is to show that the solutions really arrive at one of the two kinds of final states, and that the final state depends continously on the initial condition. For this, Conley constructed an isolating block. Isolating blocks were developed by Conley and Easton as a way to defining a topological index for invariant sets [3, 4, 6]. Among their useful properties is the fact for initial conditions which leave the isolating block, the amount of time required to leave depends continuously on initial conditions. It follows that the location of the exit point also varies continuously. In [2] Conley constructs an isolating block in a manifold of fixed Jacobi constant and uses it to justify the shooting argument outlined above.

In this paper, a figure-eight orbit will be found by shooting from the Eulerian configuration to the isosceles configurations. More precisely, initial conditions in phase space orthogonal to the set whose configuration is Eulerian will be followed under the flow and shown to meet the set whose configuration is isosceles orthogonally. This involves a higher-dimensional version of the usual shooting argument. It turns out that topologically, the problem is to show that a two-dimensional surface of initial conditions can be followed under the flow to meet another two-dimensional surface inside a four-dimensional ambient space. The proof uses a linking argument. Another example of a multidimensional shooting argument based on isolating blocks and linking can be found in [1].

The concept of isolating block is related to earlier ideas of Wazewski [11]. It is possible to get the crucial property of continous exit times under weaker assumptions than are needed for the topological index theory. For example, whereas isolating blocks are always compact, Wazewski sets need not be. In this paper the proof will be based on the construction of a Wazewski set, rather than an isolating block.

A computer-assisted existence proof based on shooting can be found in [7]. In that paper, the computations which are carried out near the actual figure-eight orbit suffice to establish local continuity of an appropriate Poincaré map and to find the required symmetric orbit. The method used in the present paper does not rely on numerical computations and provides a continous flow-defined map defined on a large region of phase space.

Since existence of a figure-eight solution was already known, some justification for a new proof may be required. The proof presented here seems interesting for several reasons. First it connects the figure-eight with a number of well-known features of the three-body problem, including double and triple collisions, central configurations and homothetic solutions. These are used in studying the boundary behavior of the map in the shooting argument (particularly in the proof of lemma 5). Second it locates a figure-eight solution in a smaller region of phase space than the variational proof – namely, in the Wazewski set $\mathcal{W}$ used for the proof. This shows that the solution moves monotonically in shape space, a fact which is not obvious from the other proof. Finally, there may be topological proofs similar to this one for the existence of other symmetric periodic solutions which have been discovered numerically but for which the variational methods fail.

2. Equation of Motion and Reduction

Consider the planar three-body problem with equal masses $m_1 = m_2 = m_3 = 1$. Let the positions be $q_i \in \mathbf{R}^2$ and the velocities be $v_i = \dot{q}_i \in \mathbf{R}^2$. The Newton's laws of motion are the Euler-Lagrange equation of the Lagrangian

$$L = \tfrac{1}{2}K + U \tag{1}$$

where

$$\begin{aligned} K &= |v_1|^2 + |v_2|^2 + |v_3|^2 \\ U &= \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{23}}. \end{aligned} \tag{2}$$

Here $r_{ij} = |q_i - q_j|$ denotes the distance between the i-th and j-th masses. The total energy of the system is constant:

$$\tfrac{1}{2}K - U = h.$$

Assume without loss of generality that total momentum is zero and that the center of mass is at the origin, i.e.,

$$v_1 + v_2 + v_3 = q_1 + q_2 + q_3 = 0.$$

Introduce Jacobi variables

$$\xi_1 = q_2 - q_1 \qquad \xi_2 = q_3 - \tfrac{1}{2}(q_1 + q_2)$$

and their velocities $\eta_i = \dot{\xi}_i$. Then the equations of motion are given by a Lagrangian of the same form (1) where now

$$\begin{aligned} K &= \tfrac{1}{2}|\eta_1|^2 + \tfrac{2}{3}|\eta_2|^2 \\ U &= \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{23}}. \end{aligned} \tag{3}$$

The mutual distances are given by

$$
\begin{aligned}
r_{12} &= |\xi_1| \\
r_{13} &= |\xi_2 + \tfrac{1}{2}\xi_1| \\
r_{23} &= |\xi_2 - \tfrac{1}{2}\xi_1|.
\end{aligned}
\tag{4}
$$

The use of Jacobi coordinates eliminates the translational symmetry of the problem and reduces the number of degrees of freedom from 6 to 4. The next step is the elimination of the rotational symmetry to reduce from 4 to 3 degrees of freedom. This is accomplished by fixing the angular momentum and working in a quotient space. When the angular momentum is zero, there is a particularly elegant way to accomplish this reduction. The discussion below follows Montgomery [9].

Define new variables r, w_1, w_2, w_3 via:

$$
\begin{aligned}
r^2 &= \tfrac{1}{2}|\xi_1|^2 + \tfrac{2}{3}|\xi_2|^2 \\
w_1 &= \tfrac{1}{4}|\xi_1|^2 - \tfrac{1}{3}|\xi_2|^2 \\
w_2 + i\, w_3 &= \tfrac{1}{\sqrt{3}}\, \xi_1 \bar{\xi}_2
\end{aligned}
\tag{5}
$$

where in the last line, vectors in $\mathbf{R}^2$ are identified with complex numbers. It is easy to check that the new variables satisfy the relation:

$$w_1^2 + w_2^2 + w_3^2 = \tfrac{1}{4}I^2 = \tfrac{1}{4}r^4.$$

The quantity $I = r^2$ is the moment of inertia, a convenient measure of the overall size of the triangle formed by the three bodies. The shape of the triangle can be represented by the normalized vector $S = 2w/r^2$ which lies in the unit sphere $\mathbf{S}^2$, which will be called the shape sphere.

Using the fact that the angular momentum is zero, it can be shown [9] that the variables r, S satisfy the Euler-Lagrange equations of a reduced Lagrangian of the form

$$L = \tfrac{1}{2}K + \tfrac{1}{r}W(S) \tag{6}$$

where:

$$
\begin{aligned}
K &= \dot{r}^2 + \tfrac{1}{4}r^2|\dot{S}|^2 \\
W &= \frac{1}{\rho_{12}} + \frac{1}{\rho_{13}} + \frac{1}{\rho_{23}}.
\end{aligned}
\tag{7}
$$

Here $|\_|$ is the Euclidean metric on the unit sphere and $\rho_{ij} = r_{ij}/r$ is the normalized interparticle distance, i.e., the interparticle distance after scaling the configuration to have moment of inertia 1. After some computation using (4) and (5) one finds

$$
\begin{aligned}
\rho_{12}^2 &= 1 + s_1 \\
\rho_{13}^2 &= 1 - \tfrac{1}{2}s_1 + \tfrac{\sqrt{3}}{2}s_2 \\
\rho_{23}^2 &= 1 - \tfrac{1}{2}s_1 - \tfrac{\sqrt{3}}{2}s_2
\end{aligned}
\tag{8}
$$

where $S = (s_1, s_2, s_3)$.

Figure 2 shows a contour plot of the shape potential W. The equator of the shape sphere is represented by $s_3 = w_3 = 0$ which means that the vectors ξ_1, ξ_2 are parallel and so the three bodies are collinear. There are three saddle points of W on the equator corresponding to the three collinear central configurations. For example, at the point $S = (1, 0, 0)$ the scaled distances are $\rho_{12} = \sqrt{2}$ and $\rho_{13} = \rho_{23} = 1/\sqrt{2}$

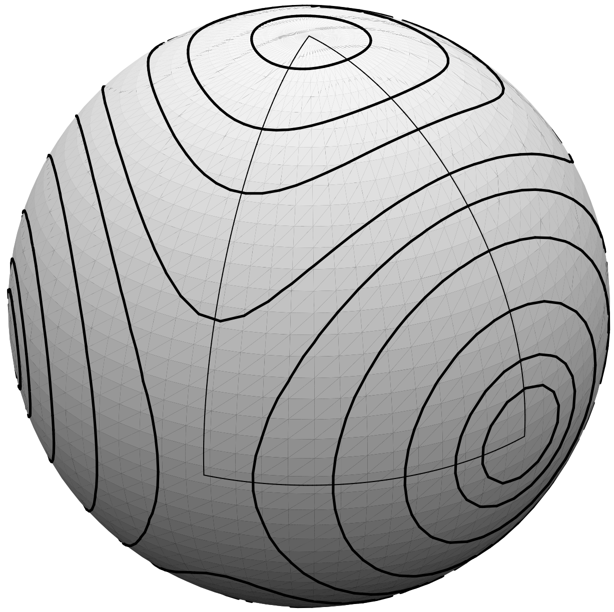

FIGURE 2. Contour plot of the shape potential W and the fundamental region $\mathcal{F}$. The top corner of $\mathcal{F}$ (the north pole) is the equilateral triangle configuration l. The bottom edge (on the equator) consist of collinear configurations. The left endpoint of this edge is the Eulerian central configuration e while the right endpoint is the double collision singularity.

so mass m_3 is at the midpoint of the line segment between masses m_1, m_2. This shape is the collinear central configuration in the equal mass case. In addition to the saddle points, there are three singular points of W along the equator at the binary collision shapes where $\rho_{ij} = 0$. One has $\rho_{12} = 0$ at $S = (-1, 0, 0)$ while $\rho_{13} = 0$ and $\rho_{23} = 0$ at $S = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ and $S = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ respectively. The north and south poles of the sphere represent the two equilateral triangle shapes, which are also central configurations.

Because of the equal masses the shape potential W has a 12-fold symmetry. In fact, a triangular region $\mathcal{F}$ as in figure 2 can serve as a fundamental region in the sense that the shape sphere can be tiled by 12 reflected and rotated copies of $\mathcal{F}$ while preserving W. The fundamental region $\mathcal{F}$ will be the spherical triangle with vertices at $S = (1, 0, 0)$, the collinear central configuration, $S = (0, 0, 1)$, one of the equilateral central configurations, and $S = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, the double collision where $\rho_{23} = 0$. The edges of $\mathcal{F}$ consist of two meridional arcs and an arc of the equator. The left meridian in figure 2 consists of isosceles configurations with m_3 on the axis of symmetry. Similarly the right meridian is made up of isosceles configurations with m_1 on the axis of symmetry. The equatorial arc on the bottom represents collinear configurations.

In Chenciner and Montgomery's original existence proof for the figure-eight orbit, this symmetry plays an important role. One may parametrize their orbit in such a way that its projection to the shape sphere begins at the collinear central configuration $S = (1, 0, 0)$ (the lower left vertex of $\mathcal{F}$). Then, in the first 1/12 of its period, it moves across the fundamental region $\mathcal{F}$ to meet the right isosceles meridian orthogonally. The rest of the orbit can be obtained by symmetry. The proof presented here will show by a topological shooting argument that there is an orbit segment whose projection to the shape sphere crosses $\mathcal{F}$ in this way and which can be continued by the 12-fold symmetry to a periodic orbit.

Two different systems of coordinates will be used on the shape sphere. The first set of *stereographic* coordinates are derived by stereographic projection from the south pole. With $S = (s_1, s_2, s_3)$ as above, let $z = (z_1, z_2)$ where

$$z_1 = \frac{s_1}{1 + s_3} \qquad z_2 = \frac{s_2}{1 + s_3} \qquad \zeta_1 = \dot{z}_1 \qquad \zeta_2 = \dot{z}_2.$$

The inverse formulas are

$$s_1 = \frac{2z_1}{1 + |z|^2} \qquad s_2 = \frac{2z_2}{1 + |z|^2} \qquad s_3 = \frac{1 - |z|^2}{1 + |z|^2}.$$

The Lagrangian is (6) where now

$$\begin{aligned}
K &= \dot{r}^2 + \frac{r^2(\dot{z}_1^2 + \dot{z}_2^2)}{(1 + |z|^2)^2} \\
W &= \frac{1}{\rho_{12}} + \frac{1}{\rho_{13}} + \frac{1}{\rho_{23}} \\
\rho_{12}^2 &= \frac{(z_1 + 1)^2 + z_2^2}{1 + |z|^2} \\
\rho_{13}^2 &= \frac{(z_1 - \frac{1}{2})^2 + (z_2 + \frac{\sqrt{3}}{2})^2}{1 + |z|^2} \\
\rho_{23}^2 &= \frac{(z_1 - \frac{1}{2})^2 + (z_2 - \frac{\sqrt{3}}{2})^2}{1 + |z|^2}.
\end{aligned} \tag{9}$$

The other set of *stereographic polar* coordinates are the polar coordinates in the z-plane. Let $z = (s\cos\theta, s\sin\theta)$ where $s = |z|$. Then the Lagrangian is (6) with

$$\begin{aligned}
K &= \dot{r}^2 + \frac{r^2(\dot{s}^2 + s^2\dot{\theta}^2)}{(1 + s^2)^2} \\
W &= \frac{1}{\rho_{12}} + \frac{1}{\rho_{13}} + \frac{1}{\rho_{23}} \\
\rho_{12}^2 &= \frac{1 + s^2 + 2s\cos\theta}{1 + s^2} \\
\rho_{13}^2 &= \frac{1 + s^2 - s\cos\theta + \sqrt{3}s\sin\theta}{1 + s^2} \\
\rho_{23}^2 &= \frac{1 + s^2 - s\cos\theta - \sqrt{3}s\sin\theta}{1 + s^2}.
\end{aligned} \tag{10}$$

These Lagrangian systems represent the zero angular momentum three-body problem reduced to 3 degrees of freedom by elimination of all the symmetries. The differential equation are the usual second-order Euler-Lagrange equations which can be written as a first-order system in phase space. One final improvement is to blow-up the triple collision singularity at $r = 0$ by introducing the time rescaling $' = r^{\frac{3}{2}}\,\dot{}$ and the variable $v = r'/r$ [8]. The result is the following first-order system:

$$
\begin{aligned}
r' &= vr \\
v' &= W(z) - \tfrac{1}{2}v^2 + 2rh \\
z_1' &= \zeta_1 \\
z_2' &= \zeta_2 \\
\zeta_1' &= (1+|z|^2)^2\, W_{z_1} - \tfrac{1}{2}v\zeta_1 + \frac{2}{1+|z|^2}\left(|\zeta|^2 z_1 - 2\zeta_2(z_1\zeta_2 - z_2\zeta_1)\right) \\
\zeta_2' &= (1+|z|^2)^2\, W_{z_2} - \tfrac{1}{2}v\zeta_2 + \frac{2}{1+|z|^2}\left(|\zeta|^2 z_2 + 2\zeta_1(z_1\zeta_2 - z_2\zeta_1)\right).
\end{aligned}
\tag{11}
$$

with energy equation:

$$
\tfrac{1}{2}v^2 + \tfrac{1}{2}\frac{|\zeta|^2}{(1+|z|^2)^2} - W(z) = rh \tag{12}
$$

Note that $\{r = 0\}$ is now an invariant set for the flow, called the triple collision manifold. The differential equations are still singular due to the double collisions when $\rho_{ij} = 0$. However, in the argument below, it is possible to avoid these.

The corresponding equations in stereographic polar coordinates are:

$$
\begin{aligned}
r' &= vr \\
v' &= W(s,\theta) - \tfrac{1}{2}v^2 + 2rh \\
s' &= \sigma \\
\theta' &= \tau \\
\sigma' &= (1+s^2)^2\, W_s - \tfrac{1}{2}v\sigma + \frac{2s\sigma^2}{1+s^2} + \frac{s(1-s^2)\tau^2}{1+s^2} \\
s^2\tau' &= (1+s^2)^2\, W_\theta - \tfrac{1}{2}vs^2\tau - \frac{2s\sigma\tau}{1+s^2}.
\end{aligned}
\tag{13}
$$

with energy equation:

$$
\tfrac{1}{2}v^2 + \tfrac{1}{2}\frac{\sigma^2 + s^2\tau^2}{(1+s^2)^2} - W(s,\theta) = rh \tag{14}
$$

Figures 3 and 4 show the potential W in stereographic and stereographic polar coordinates, respectively.

3. A Wazewski Set

Consider the system (11) on a manifold of fixed energy $h < 0$. Due to scaling symmetry, one may assume without loss of generality that $h = -1$. The energy equation (12) shows that the energy manifold is five-dimensional and is a graph over its projection to the (v, z, ζ)-space. The goal of this section is to construct a Wazewski set in this five-dimensional manifold.

A Wazewski set for a flow $\phi_t(x)$ on a topological space X is a subset $\mathcal{W} \subset X$ satisfying technical hypotheses which guarantee that the time required to exit $\mathcal{W}$ depends continuously on initial conditions [11, 3]. To formulate these hypotheses,

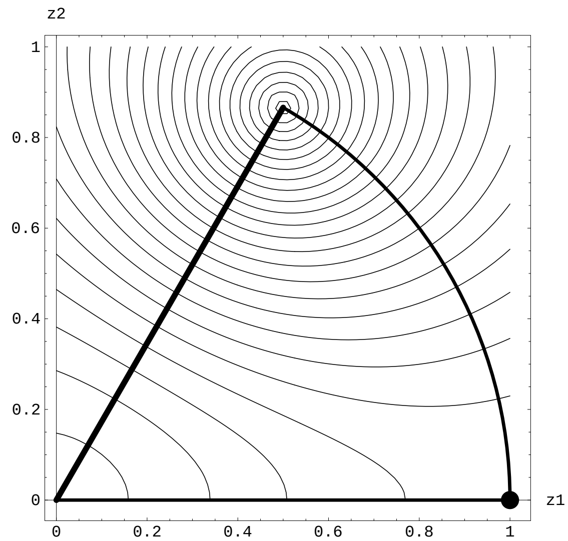

FIGURE 3. Contour plot of the shape potential W and the fundamental region $\mathcal{F}$ in stereographic coordinates. The dot represents collinear central configuration and the opposite bold edge of $\mathcal{F}$ represents the isosceles configurations which will enter into the shooting argument. The top corner of $\mathcal{F}$ is the double collision point $z = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

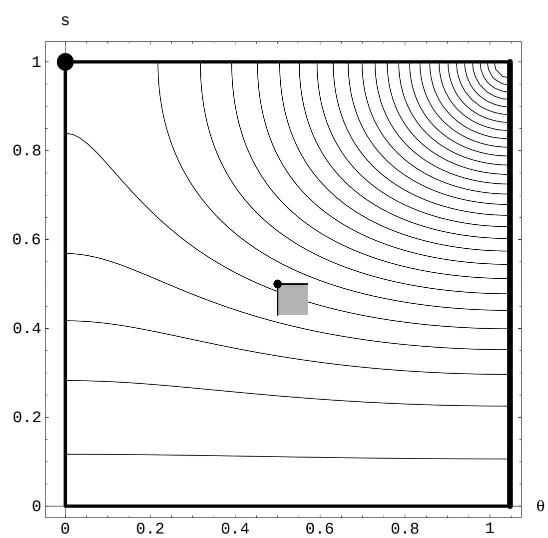

FIGURE 4. Contour plot of the shape potential W and the fundamental region $\mathcal{F}$ in stereographic polar coordinates. The fundamental region $\mathcal{F}$ is the square $0 \leq \theta \leq \pi/3$, $0 \leq s \leq 1$. The south-east velocity cone field defining the Wazewski set is shown in gray.

let $\mathcal{W}^0$ be the set of points in $\mathcal{W}$ which eventually leave $\mathcal{W}$ in forward time, and let $\mathcal{E}$ be the set of points which exit immediately:

$$\mathcal{W}^0 = \{x \in \mathcal{W} : \exists t > 0, \phi_t(x) \notin \mathcal{W}\}$$
$$\mathcal{E} = \{x \in \mathcal{W} : \forall t > 0, \phi_{[0,t)}(x) \not\subset \mathcal{W}\}.$$

Clearly, $\mathcal{E} \subset \mathcal{W}^0$. Given $x \in \mathcal{W}^0$ define the *exit time*

$$\tau(x) = \sup\{t \geq 0 : \phi_{[0,t]}(x) \subset \mathcal{W}\}.$$

Note that $\tau(x) = 0$ if and only if $x \in \mathcal{E}$.

The appropriate hypotheses which guarantee continuity of τ are [3]:

a. If $x \in \mathcal{W}$ and $\phi_{[0,t]}(x) \subset \overline{\mathcal{W}}$, then $\phi_{[0,t]}(x) \subset \mathcal{W}$
b. $\mathcal{E}$ is a relatively closed subset of $\mathcal{W}^0$

The choice of the set $\mathcal{W}$ is motivated by the shooting argument outlined above. Recall the spherical triangle $\mathcal{F}$, the fundamental region for the potential. Roughly speaking, the Wazewski set consists of the points in phase space whose shapes lie in $\mathcal{F}$ and whose velocities lie in a certain cone. In stereographic polar coordinates, the velocity cone is given by

$$s' = \sigma \leq 0 \qquad \theta' = \tau \geq 0.$$

A typical element of the is cone field is shaded gray in figure 4. Thus the solutions in $\mathcal{W}$ will be moving south-east across the figure. Because of the polar coordinate singularity at $s = 0$, the actual definition of $\mathcal{W}$ will be given in ordinary stereographic coordinates, where the velocity conditions become

$$\begin{array}{lll} z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2 \leq 0 & z \wedge \zeta = z_1\zeta_2 - z_2\zeta_1 \geq 0 & \text{if } (z_1, z_2) \neq (0,0) \\ (1,0) \cdot \zeta = \zeta_1 \leq 0 & (1,\sqrt{3}) \wedge \zeta = \zeta_2 - \sqrt{3}\zeta_1 \geq 0 & \text{if } (z_1, z_2) = (0,0). \end{array} \tag{15}$$

This includes a special definition for the origin, which is taken to be the union of all of the velocity cones at nearby points $z \in \mathcal{F}$, $z \neq (0,0)$. Let

$$\overline{\mathcal{W}} = \{(r, v, z, \zeta) : (12) \text{ holds}, r \geq 0, z \in \mathcal{F}, \zeta \text{ satisfies } (15)\}. \tag{16}$$

Because of the way the cone at $z = (0,0)$ was handled, $\overline{\mathcal{W}}$ is a closed subset of $\mathbf{R}^6$.

For technical reasons, certain parts of the set $\overline{\mathcal{W}}$ must be excluded from the Wazewski set. First, there is the double collision singularity at the vertex $z = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let

$$\mathcal{D} = \{(r, v, z, \zeta) : z = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\} = \{(r, v, s, \theta, \sigma, \tau) : s = 1, \theta = \tfrac{\pi}{3}\}.$$

Next, there is the collinear invariant manifold, which is given by

$$\mathcal{C} = \{(r, v, z, \zeta) : |z| = 1, z_1\zeta_1 + z_2\zeta_2 = 0\} = \{(r, v, s, \theta, \sigma, \tau) : s = 1, s' = \sigma = 0\}.$$

On the one hand, these orbits will have double collisions; on the other hand, their exiting behavior is somewhat different from that of the nearby, non-collinear orbits. Finally there is the set

$$\mathcal{I}_1 = \{(r, v, z, \zeta) : z_2 - \sqrt{3}z_1 = \zeta_2 - \sqrt{3}\zeta_1 = 0\}$$
$$\mathcal{I}_1 \cap \{s > 0\} = \{(r, v, s, \theta, \sigma, \tau) : 0 < s \leq 1, \theta = \tfrac{\pi}{3}, \tau = 0\}.$$

This is an invariant manifold of isosceles motions (with mass m_1 moving on the symmetry axis of the isosceles triangle). There is another invariant isosceles manifold $\mathcal{I}_3$ given by $\theta = \tau = 0$ which will play a role later on. However, it is convenient and feasible to keep $\mathcal{I}_3$ in the Wazewski set.

With these definitions in place, the Wazewski set is

$$\mathcal{W} = \overline{\mathcal{W}} \setminus \mathcal{C} \cup \mathcal{D} \cup \mathcal{I}_1.$$

Note that $\overline{\mathcal{W}}$ is the closure of $\mathcal{W}$ in $\mathbf{R}^6$ as the notation suggests. Also $\mathcal{W}$ is open in $\overline{\mathcal{W}}$.

The rest of this section is devoted to proving

Theorem 1. *$\mathcal{W}$ is a Wazewski set for the flow on the constant energy manifold.*

First, note that any initial condition in $\mathcal{W}$ defines a solution which continues to exist as long as it remains in $\mathcal{W}$. To see this recall that it is a well-known property of the three-body problem that a solution which fails to exist for all time must end in collision [10]. Since we have blown-up the triple collision, the only possibility for a singularity is the double collision at $(s, \theta) = (1, \frac{\pi}{3})$. If the initial value $s_0 < 1$ then since $s(t)$ is non-increasing for orbits in $\mathcal{W}$, it is impossible to reach the double collision. Similarly if $s_0 = 1$ but $\sigma_0 < 0$, then $s(t) < 1$ for all $t > 0$ and again, collision cannot occur. The only remaining case, $s_0 = 1, \sigma_0 = 0$ is excluded as a point of $\mathcal{C}$.

To verify Wazewski property **a** one must check that whenever $x \in \mathcal{W}$ and the closed orbit segment $\phi_{[0,t]}(x)$ is contained in the closure $\overline{\mathcal{W}}$, then $\phi_{[0,t]}(x)$ is actually contained in $\mathcal{W}$ itself. Assume for the sake of contradiction, that t is the smallest time such $\phi_t(x) \in \overline{\mathcal{W}} \setminus \mathcal{W}$. In particular, $t > 0$ since $x \in \mathcal{W}$ by assumption. Since $\overline{\mathcal{W}} \setminus \mathcal{W} \subset \mathcal{C} \cup \mathcal{D} \cup \mathcal{I}_1$, it suffices to show that an orbit segment beginning in $\mathcal{W}$ cannot enter $\mathcal{C} \cup \mathcal{D} \cup \mathcal{I}_1$. Now the argument of the last paragraph shows that for the orbit segment under consideration, $s(t) < 1$ so $\phi_t(x) \notin \mathcal{C} \cup \mathcal{D}$. Also, $\phi_t(x) \notin \mathcal{I}_1$ since $x \notin \mathcal{I}_1$ and $\mathcal{I}_1$ is an invariant set. Thus property **a** holds for $\mathcal{W}$.

To check property **b**, one must first identify the subsets $\mathcal{W}^0, \mathcal{E}$. It turns out that all of the solutions beginning in $\mathcal{W}$ eventually leave, except those which converge to one of the equilibrium points on the triple collision manifold. There are exactly four equilibrium points in $\mathcal{W}$. In (r, v, z, ζ) coordinates, they are

$$E_\pm = (0, \pm\sqrt{2W(e)}, e, 0) \qquad L_\pm = (0, \pm\sqrt{2W(l)}, l, 0)$$

where e, l are the Eulerian and Lagrangian central configurations, with z-coordinates:

$$e = (1, 0) \qquad l = (0, 0).$$

These are just the critical points of the potential $W(z)$ in $\mathcal{F}$ (see figure 3). It is known that these are hyperbolic equilibria for the differential equation (11). It turns out that parts of the stable manifolds of $L_\pm$ lie in $\mathcal{W}$ and the corresponding solutions may never exit. Define the immediate stable manifolds:

$$W_0^s(L_\pm) = \{x \in \mathcal{W} : \phi_t(x) \in \mathcal{W} \text{ for all } t \geq 0, \phi_t(x) \to L_\pm \text{ as } t \to \infty\}.$$

Then

Lemma 1. $\mathcal{W}^0 = \mathcal{W} \setminus W_0^s(L_\pm)$.

Proof. Consider a solution $\phi_t(x)$ which remains in $\mathcal{W}$ for all $t \geq 0$. On the energy manifold, one can use (12) to eliminate r from the differential equations to obtain a five-dimensional differential equation for (v, z, ζ) or $(v, s, \theta, \sigma, \tau)$. The monotonicity of $s(t)$ and $\theta(t)$ and the compactness of $\mathcal{F}$ imply that $z_\infty = \lim_{t\to\infty} z(t)$ exists and is a point of $\mathcal{F}$. Since $s(t) < 1$ for $t > 0$ and is non-increasing, it follows that $|z_\infty| < 1$. In particular, z_∞ cannot be the Eulerian central configuration or the double collision.

The energy equation gives a bound

$$v^2 + \frac{|\zeta|^2}{(1+|z|^2)^2} \leq 2W(z).$$

Thus the omega limit set $\omega(x)$ is nonempty and is contained in the compact set

$$z = z_\infty \qquad v^2 + \frac{|\zeta|^2}{(1+|z_\infty|^2)^2} \leq 2W(z_\infty).$$

Any invariant subset of this form must have $\zeta = \zeta' = 0$ and then (11) shows that $W_{z_1} = W_{z_2} = 0$, i.e., z_∞ is a critical point of the shape potential. The only critical point with $|z_\infty| < 1$ is the Lagrangian central configuration l so $z_\infty = l$.

Now the line segment $z = l, \zeta = 0,\ v^2 \leq 2W(l)$ is the Lagrangian homothetic solution connecting the restpoint L_+ to L_-. It follows that $\phi_t(x)$ must converge to one of these restpoints, as claimed. QED

To find the immediate exit set $\mathcal{E}$ one must examine the boundary points of $\mathcal{W}$. It turns out that there are essentially just two ways to exit $\mathcal{W}$ – either the orbit reaches an isosceles configuration (with m_1 on the symmetry axis) or the stereographic polar radius stops decreasing and begins to increase again. Corresponding to these two modes of exiting, it is convenient to distinguish two subsets of the boundary. Let $x = (r, v, z, \zeta)$ and let

$$\begin{aligned} \mathcal{B}_1 =& \{x \in \mathcal{W} : z_2 - \sqrt{3} z_1 = 0\} \\ \mathcal{B}_2 =& \{x \in \mathcal{W} : z_1\zeta_1 + z_2\zeta_2 = 0\}. \end{aligned}$$

The parts of these sets with $|z| > 0$ can be described more intuitively in stereographic polar coordinates $x = (r, v, s, \theta, \sigma, \tau)$ as

$$\begin{aligned} \mathcal{B}_1 \cap \{s > 0\} &= \{x \in \mathcal{W} : \theta = \tfrac{\pi}{3}\} \\ \mathcal{B}_2 \cap \{s > 0\} &= \{x \in \mathcal{W} : s' = \sigma = 0\}. \end{aligned}$$

When $z = (0,0)$ the conditions on the velocities reduce to those required for membership in $\mathcal{W}$. These definitions make it clear that $\mathcal{B}_1$ and $\mathcal{B}_2$ are relatively closed subsets of $\mathcal{W}$, hence also of $\mathcal{W}_0$. So the following lemma completes the verification of property **b** and the proof the theorem 1.

Lemma 2. *The immediate exit set of* $\mathcal{W}$ *is* $\mathcal{E} = \mathcal{B}_1 \cup \mathcal{B}_2$.

Proof. By definition, $\mathcal{B}_1, \mathcal{B}_2$ each contain $\{x \in \mathcal{W} : z = (0,0)\}$. Note that for any such x, the shape velocity $\zeta \neq (0,0)$ since $z = \zeta = 0$ is part of the isosceles invariant manifold $\mathcal{I}_1$ which was excluded from $\mathcal{W}$. Furthermore, by definition of the velocity cone when $z = (0,0)$, ζ points out of $\mathcal{F}$ and so these points x are indeed in $\mathcal{E}$. For the rest of the proof one may assume $z \neq (0,0)$.

Let $x = (r, v, z, \zeta) \in \mathcal{B}_1$ with $z \neq (0,0)$. Using stereographic polar coordinates, one has $\theta = \frac{\pi}{3}$ and since the isosceles set $\mathcal{I}_1$ has been excluded $\theta' = \tau > 0$. Thus x is an immediate exit point and $\mathcal{B}_1 \subset \mathcal{E}$.

Next consider a point $x = (r, v, s, \theta, \sigma, \tau) \in \mathcal{B}_2$ with $s > 0$. Since $\sigma = 0$, (20) gives

$$\sigma' = (1+s^2)^2 W_s + \frac{s(1-s^2)\tau^2}{1+s^2} \geq (1+s^2)^2 W_s.$$

Now it is easy to check that $W_s \geq 0$ in $\mathcal{F}$ with equality only if $s = 0, 1$. One has $s > 0$ by assumption and $s < 1$ since the set $s = 1, \sigma = 0$ is the excluded collinear invariant manifold $\mathcal{C}$. Therefore $\sigma' > 0$ for all points of $\mathcal{B}_2$ with $s > 0$. This shows $\mathcal{B}_2 \subset \mathcal{E}$.

To complete the proof, it remains to show that there are no other immediate exit points. Suppose, for the sake of contradiction, that $x \in \mathcal{W}$ is an immediate exit point which is not in $\mathcal{B}_1 \cup \mathcal{B}_2$. Only the case $s > 0$ needs to be considered since the points with $s = 0$ are in $\mathcal{B}_1 \cup \mathcal{B}_2$. By definition of $\mathcal{E}$ there is a sequence of times $t_n > 0$, $t_n \to 0$ such that $\phi_{t_n}(x) \notin \mathcal{W}$. By Wazewski property **a**, one may assume $\phi_{t_n}(x) \notin \overline{\mathcal{W}}$.

From the definition (16), there are four other conceivable ways to exit $\overline{\mathcal{W}}$, each of which must be ruled out. First, one could exit by having $r = 0$ but $r(t_n) < 0$ for a sequence of positive times $t_n \to 0$. However, this is impossible because $\{r = 0\}$ is the invariant triple collision manifold.

Next, one might have $s = 1$ and $s(t_n) > 1$ for small positive times as above. Clearly this implies $s' = \sigma \geq 0$, but all points $x \in \mathcal{W}$ with $s = 1$ have $\sigma < 0$ so this is also impossible.

If $\theta = 0$ and $\theta(t_n) < 0$, one would have $\theta' = \tau = 0$. However, the equations $\theta = \tau = 0$ define the invariant manifold of isosceles orbits $\mathcal{I}_3$ (with m_3 on the axis of symmetry). Therefore $\theta(t)$ remains constant rather than decreasing.

Finally, if $\tau = 0$ and $\tau(t_n) < 0$, then it must be that $s^2\tau' = (1 + s^2)^2 W_\theta \leq 0$. Now it can be shown that $W_\theta \geq 0$ in $\mathcal{F}$ with equality only when $\theta = 0, \pi/3$. The case $\theta = \tau = 0$ has just been discussed while $\theta = \pi/3, \tau = 0$ defines the invariant isosceles manifold $\mathcal{I}_1$ which has been excluded from $\mathcal{W}$. This completes the proof. QED

4. The Shooting Argument

The idea is to find the first twelfth of the figure-eight orbit by shooting from the collinear central configuration with m_3 in the middle to the set of isosceles configurations with m_1 on the symmetry axis (see figure 1). A certain set of initial conditions $\mathcal{S}_0$ with $z = (1, 0)$, the collinear central configuration, will be followed under the flow until they exit $\mathcal{W}$ and it will be shown that at least one of the resulting exit points lies in a certain target set $\mathcal{T}$ with isosceles configurations, $z_2 = \sqrt{3}z_1$. In figure 3, the projection of the first twelfth of the orbit will move from the dot at one vertex of the fundamental triangle to the opposite edge. In figure 4, it will move the upper left corner to the right edge. Moreover, appropriate orthogonality conditions on the velocities must be met.

More precisely, let

$$\mathcal{S} = \{(r, v, z, \zeta) \in \mathcal{W} : z = (1, 0), v = 0\}$$
$$\mathcal{T} = \{(r, v, z, \zeta) \in \mathcal{W} : z_2 - \sqrt{3}z_1 = \zeta_1 + \sqrt{3}\zeta_2 = v = 0\}.$$

Using stereographic polar coordinates one has

$$\mathcal{S} = \{(r, v, s, \theta, \sigma, \tau) \in \mathcal{W} : s = 1, \theta = v = 0\}$$
$$\mathcal{T} \cap \{s > 0\} = \{(r, v, s, \theta, \sigma, \tau) \in \mathcal{W} : \theta = \tfrac{\pi}{3}, \sigma = v = 0\}.$$

Imposing the equations $v = 0$ in $\mathcal{S}$ and $\sigma = v = 0$ in $\mathcal{T}$ guarantees that any orbit segment beginning in $\mathcal{S}$ and ending in $\mathcal{T}$ can be extended using the 12-fold symmetry to a periodic solution. The proof actually uses a proper subset $\mathcal{S}_0 \subset \mathcal{S}$ which will be described later.

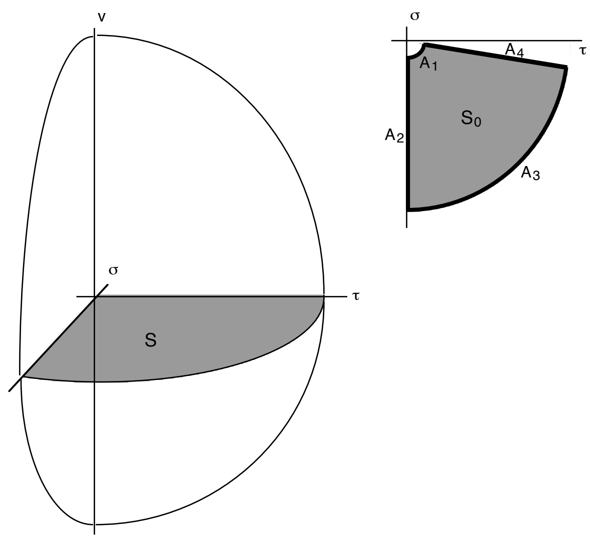

FIGURE 5. Velocity fiber over the Eulerian central configuration. $\mathcal{S}_0 \subset \mathcal{S}$ is the initial set for the shooting argument.

It is possible to visualize $\mathcal{S}, \mathcal{T}$ and to see how $\mathcal{T}$ is embedded in the immediate exit set $\mathcal{E}$. There are six variables, but r will always be eliminated using the energy relation. Since points of $\mathcal{S}$ all have the same shape, it is only necessary to draw the velocity variables (v, ζ_1, ζ_2) or (v, σ, τ). Using the polar variables, (21) shows that the projection of the energy manifold is the solid region where

$$v^2 + \frac{\sigma^2 + s^2\tau^2}{(1+s^2)^2} \leq 2W(s, \theta).$$

For $(s, \theta) = (1, 0)$ this is a spheroid. The inequalities $\sigma \leq 0, \tau \geq 0$ defining the velocity cone cut out a quarter of the spheroid, as shown in figure 5. The set $\mathcal{S}$ is the slice $v = 0$. Thus $\mathcal{S}$ is a two-dimensional surface. It is a quarter circle, homeomorphic to a disk.

The initial set $\mathcal{S}$ will be shrunk to a more manageable subset $\mathcal{S}_0$ as shown in figure 5. Let

$$\mathcal{S}_0 = \{(r, v, s, \theta, \sigma, \tau) \in \mathcal{W} : s = 1, \theta = v = 0, \sigma^2 + \tau^2 \geq c_1, \sigma + c_2\tau \leq 0\} \subset \mathcal{S}.$$

This moves the initial set away from the Eulerian homothetic orbit at $\sigma = \tau = 0$ and away from the collinear invariant manifold $\mathcal{C}$ (where $s = 1, \sigma = 0$) which was deleted from $\mathcal{W}$. For later reference the four boundary arcs of $\mathcal{S}_0$ have been labelled $\mathcal{A}_1, \ldots \mathcal{A}_4$ in the figure.

The target set $\mathcal{T}$ is also homeomorphic to a two-dimensional disk. It will be important for the proof to know how $\mathcal{T}$ is embedded in the immediate exit set $\mathcal{E}$. Since the latter is four-dimensional, a projection which preserves the relevant topological properties will be used. Since the variable τ plays a lesser role in the argument, the projections will be along this axis.

First consider the set $\mathcal{B}_1$ which forms part of the immediate exit set. Since $\theta = \frac{\pi}{3}$ is constant, one can use variables (v, s, σ, τ) to parametrize it. Projecting to (v, s, σ)

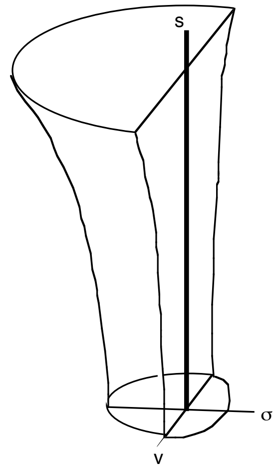

FIGURE 6. Projection of $\mathcal{B}_1$ which forms half of the immediate exit set $\mathcal{E}$. The s-axis runs vertically from $s = 0$ to $s = 1$. The diameter (σ, v)-directions approaches ∞ as $s \to 1$. The target set T is shown as a bold line segment.

space, the energy relation gives

$$v^2 + \frac{\sigma^2}{(1+s^2)^2} \le 2W(s, \tfrac{\pi}{3}).$$

The part of this solid with $\sigma \le 0$ is shown in figure 6. The outer surface of the solid corresponds to $\tau = 0$. Since one also has $\theta = \frac{\pi}{3}$, this surface is part of the isosceles manifold $\mathcal{I}_1$ which is not part of $\mathcal{W}$. Similarly, $s = 1, \theta = \frac{\pi}{3}$ is the double collision configuration which has been deleted. Over each remaining point of the solid, one should imagine a line segment of τ-values. In this picture, the projection of the target set $\mathcal{T}$ appears as the line segment $v = \sigma = 0$, $0 \le s < 1$. When $s = 0$ there is an extra flap extending away from the solid due to the larger velocity cone used there.

There is a similar picture showing the projection of $\mathcal{B}_2$ (where $\sigma = 0$) to the (v, z) space (see figure 6 where the shape is labelled by (s, θ) instead of z). Over each point $z \in \mathcal{F}$ there is a line segment of v-values: $-\sqrt{2W(z)} \le v \le \sqrt{2W(z)}$. This time, the outer surface where $\tau = 0$ is not deleted, except for the curve where it intersects the plane $z_2 = \sqrt{3}z_1$ (i.e., $\theta = \frac{\pi}{3}$). Of course over each (v, z) in the solid there is a line segment of τ-values which has been projected out. In this figure, the projection of the target $\mathcal{T}$ is the line segment $v = 0$, $z_2 = \sqrt{3}z_1$.

Finally one can identify the plane where $\sigma = 0$ in $\mathcal{B}_1$ with the plane where $\theta = \frac{\pi}{3}$ in $\mathcal{B}_2$ to obtain a projection of the entire immediate exit set $\mathcal{E}$ and the target (figure 8). Note that the target set T becomes a line segment through the middle of the solid region. In the actual, four-dimensional figure, T is a two-dimensional disk embedded in the four-dimensional immediate exit set $\mathcal{E}$. But the figure correctly depicts the essential topological relationship between T and $\mathcal{E}$ in the sense that the projection from the pair of spaces $(\mathcal{E}, T)$ onto their images in figure 8 is a homotopy

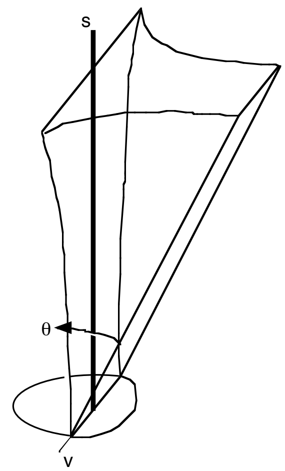

FIGURE 7. Projection of $\mathcal{B}_2$ which forms the other half of the immediate exit set $\mathcal{E}$. The variables in the projection are v and two shape variables (z_1, z_2 or s, θ) lying in the fundamental triangle $\mathcal{F}$, The target set T is shown as a bold line segment.

equivalence. This follows easily by contracting the line segments of τ values which have been projected out.

The rest of the argument will be carried out in a series of lemmas to be proved in the next section. First it will be shown that almost all of the initial points in $\mathcal{S}_0$ eventually leave $\mathcal{W}$ and so can be followed forward under the flow to the immediate exit set $\mathcal{E}$.

Lemma 3. *The initial set $\mathcal{S}_0 \subset \mathcal{W}_0$ except for those points on the isosceles edge $\theta = \tau = 0$ ($\mathcal{A}_2$ in figure 5) which are also in $W_0^s(L_-)$.*

Since $\mathcal{W}$ is a Wazewski set, there is a continuous map $F : \mathcal{W}_0 \to \mathcal{E}$ taking initial points to their exit states. This will be defined on most of $\mathcal{S}_0$. The next lemma shows that it extends to all of $\mathcal{S}_0$.

Lemma 4. *The restriction of the flow defined map across $\mathcal{W}$ can be extended to a continuous map $F : \mathcal{S}_0 \to \overline{\mathcal{E}}$ by setting $F(x) = L_-$ for any $x \in W_0^s(L_-)$.*

The main point of the shooting proof is to show that $F(\mathcal{S}_0) \cap \mathcal{T} \neq \emptyset$. This will be done by examining the behavior of F on the boundary of $\mathcal{S}_0$. This naturally divides into four arcs, $\mathcal{A}_1, \ldots, \mathcal{A}_4$ as shown in figure 5. More precisely, using stereographic polar coordinates $x = (r, v, s, \theta, \sigma, \tau)$

$$\mathcal{A}_1 = \{x \in \mathcal{S}_0 : \sigma^2 + \tau^2 = c_1\}$$
$$\mathcal{A}_2 = \{x \in \mathcal{S}_0 : \tau = 0\}$$
$$\mathcal{A}_3 = \{x \in \mathcal{S}_0 : r = 0\}$$
$$\mathcal{A}_4 = \{x \in \mathcal{S}_0 : \sigma + c_2\tau = 0\}$$

Lemma 5. *The exit behaviors of the arcs $\mathcal{A}_i$ are as follows. Points in $\mathcal{A}_1$ exit in $\mathcal{B}_2$ with $v < 0$. Points in $\mathcal{A}_2$ exit either in $\mathcal{B}_1$ at the equilateral configuration $z = 0$ or in $\mathcal{B}_2$ with $v < 0$ and $\theta = 0$. Points in $\mathcal{A}_3$ exit with $r = 0, v > 0$. Finally, points in $\mathcal{A}_4$ exit in $\mathcal{B}_2$.*

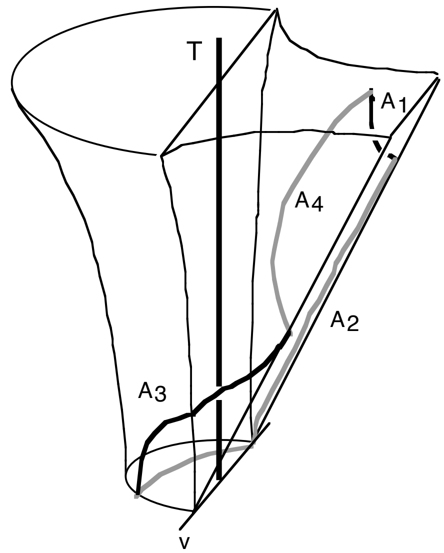

FIGURE 8. Projection of the immediate exit set $\mathcal{E}$ together with the target set T and the images under the continuous map F of the four boundary arcs, $\mathcal{A}_i$, of $\mathcal{S}_0$. The images of the four arcs (shown alternately in gray and black) link the target set.

Using these lemmas, it follows that the projection of the image under F of the boundary of $\mathcal{S}_0$ appears as shown in figure 8. In particular, the image is a closed curve which links the projection of the target set $\mathcal{T}$. Since the projection is a homotopy equivalence, it follows that $F(\mathcal{S}_0) \cap \mathcal{T} \neq \emptyset$ as required.

Using the twelve-fold symmetry one can now construct a periodic solution of the reduced differential equations. It follows as in Chenciner and Montgomery's original proof that this orbit is actually periodic for the unreduced system, i.e., the planar three-body problem. This proves the main result:

Theorem 2. *For the equal mass three-body problem with zero angular momentum and arbitrary negative energy, there exists a figure-eight periodic solution of the following type. During the first twelfth of the period, the three masses move from the collinear central configuration with mass m_3 between m_1, m_2 to an isosceles configuration with m_1 on the symmetry axis. The projection of this orbit segment to the shape sphere lies in the fundamental triangle $\mathcal{F}$ and has the property that the stereographic polar radius s is monotonically decreasing while the angle θ is monotonically increasing. The rest of the orbit is obtained by symmetry.*

5. Proofs of the lemmas.

This section contains the proofs of the lemmas 3–5.

Proof of lemma 3. It must be shown that, with the indicated exceptions, most of the points in the initial set $\mathcal{S}_0$ lie in $\mathcal{W}_0$, i.e., they eventually exit $\mathcal{W}$. By lemma 1 any points which do not exit must lie in one of the local stable manfolds $W^s_0(L_\pm)$.

First it will be shown that $\mathcal{S}_0 \cap W_0^s(L_+) = \emptyset$. It is well-known from studies of the triple collision manifold that $W^s(L_+) \subset \{r = 0\}$. Since $\{r = 0\}$ is an invariant set for the flow, it suffices to show that if $x \in S_0$ is an initial point with $r = 0$ then $x \notin W_0^s(L_+)$. Since initially $v = 0$ and since $v = \sqrt{2W(l)} = \sqrt{6}$ at L_+, it will be enough to show that the change in v along the orbit $\phi_t(x)$ satisfies $\Delta v < \sqrt{6}$. This will be done by comparing the change in v to the arclength of the projection of the solution to shape space.

For solutions with $r = 0$ we have $v' = W - \frac{1}{2}v^2 \geq 0$. On the other hand, the length of the velocity of the projected curve with respect to the kinetic energy metric satisfies

$$(\lambda')^2 = \frac{|z'|^2}{(1+|z|^2)^2} = 2W - v^2$$

where we use λ to denote arclength. Hence

$$v' = \sqrt{\tfrac{1}{2}W - \tfrac{1}{4}v^2}\,\sqrt{2W - v^2} \leq \sqrt{\tfrac{1}{2}W}\,\lambda'$$

and

$$\Delta v \leq \int_\gamma \sqrt{\tfrac{1}{2}W}\,d\lambda$$

where γ is the projected curve in the fundamental region $\mathcal{F}$.

To estimate this integral note that in stereographic polar coordinates, one has

$$d\lambda = \frac{\sqrt{ds^2 + s^2\,d\theta^2}}{1+s^2} \leq \frac{|ds|}{1+s^2} + \frac{s|d\theta|}{1+s^2} \leq \frac{|ds|}{1+s^2} + \frac{|d\theta|}{2}.$$

Along γ, s and θ are both monotonic, with s decreasing from 1 to 0 and θ increasing from 0 to at most $\frac{\pi}{3}$. Also, the shape potential $W(s,\theta)$ satifies $W_s \geq 0$ and $W_\theta \geq 0$ in $\mathcal{F}$ so increasing s to 1 or θ to $\frac{\pi}{3}$ only increases W. From these observations it follows that $\Delta v \leq I_1 + I_2$ where

$$I_1 = \int_0^1 \sqrt{\tfrac{1}{2}W(s,\tfrac{\pi}{3})}\,\frac{ds}{1+s^2} \qquad I_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sqrt{\tfrac{1}{2}W(1,\theta)}\,d\theta.$$

Using (10) one can approximate these integrals numerically with the results

$$I_1 \approx 1.34227 \qquad I_2 \approx 1.02795.$$

Hence $\Delta v \leq I_1 + I_2 \approx 2.37022 < \sqrt{6} \approx 2.44949$ as required.

It remains to show that $\mathcal{S}_0 \cap W_0^s(L_-)$ is contained in the isosceles invariant set $\mathcal{I}_3$ where $\theta = \tau = 0$. To see this, consider $x \notin \mathcal{I}_3$. Then either $\theta > 0$ or $\theta = 0, \tau > 0$. In either case, one has $\theta(t) > 0$ for small positive times t.

In $\mathcal{W}$, one has $\theta' = \tau \geq 0$. So orbits in $W_0^s(L_-)$ approach L_- in such at way that $\theta(t)$ has a limit and of course $s(t)$ decreases monotonically to 0. The differential equations (20) give

$$\tau' = \frac{(1+s^2)^2}{s^2}\,W_\theta - \tfrac{1}{2}v\tau - \frac{2\sigma\tau}{s(1+s^2)}.$$

It can be shown that $W_\theta \geq 0$ in $\mathcal{F}$ with equality only along the isosceles lines $\theta = 0, \frac{\pi}{3}$. First of all, this shows that it is not possible for a non-isosceles orbit in $W_0^s(L_-)$ to approach L_- with $\theta(t)$ constant and $\tau(t)$ identically 0. But then $\theta(t)$ must converge to its limit strictly from below. In that case, there would have to be a sequence of times $t_n \to \infty$ with $\theta'(t_n) = \tau(t_n) > 0, \theta''(t_n) = \tau'(t_n) < 0$. But since, $v(t_n) \to -\sqrt{6}$ the differential equation shows that $\tau(t_n) > 0$ implies $\tau'(t_n) > 0$, a contradiction. QED

Proof of lemma 4. Consider a point $x \in W^s_0(L_-)$. It was shown in lemma 3 that such a point must lie in the isosceles invariant manifold $\mathcal{I}_3$ but this will not be needed here. Note that the equilibrium point L_- lies on a corner of $\overline{\mathcal{W}}$ where the boundary manifolds $r = 0$, $\theta = \frac{\pi}{3}$, $\sigma = 0$, $\tau = 0$ come together (L_- itself is not part of $\mathcal{W}$ since it lies is $\mathcal{I}_1$).

The main point is that the local punctured unstable manifold $W^u(L_-) \setminus L_-$ lies outside of $\overline{\mathcal{W}}$. Indeed, a computation using (v, z, ζ) as local coordinates shows that the unstable eigenspace consists of all vectors of the form

$$(\delta v, \delta z, \delta\zeta) = (0, u, ku)$$

with $u \in \mathbf{R}^2$, and $k = \frac{1}{2}\sqrt{21 + 3\sqrt{13}} > 0$. On this subspace the quadratic form $z \cdot \zeta$ is positive definite whereas, by definition, $z \cdot \zeta \leq 0$ in $\overline{\mathcal{W}}$.

Now consider a point $y \in \mathcal{W}_0$ near the point $x \in W^s_0(L_-)$. Let a small neighborhood $\mathcal{U}$ of L_- be given and let $T > 0$ be such that $\phi_t(x) \in \mathcal{U}$ for $t \geq T$. Now the immediate exit set $\mathcal{E}$ is closed in $\mathcal{W}_0$ and it is disjoint from the compact orbit segment $\phi_{[0,T]}(x)$. It follows that if $y \in \mathcal{W}_0$ is sufficiently close to x, it cannot exit $\mathcal{W}$ before entering $\mathcal{U}$. On the other hand, after entering $\mathcal{U}$, the lambda lemma shows that $\phi_t(y)$ passes close to L_- and emerges from $\mathcal{U}$ close to $W^u(L_-)$. Since $W^u(L_-) \setminus L_-$ lies outside of $\overline{\mathcal{W}}$, it follows that $\phi_t(y)$ exits $\mathcal{W}$ somewhere in $\mathcal{U}$, i.e., close to L_-. This means that defining $F(x) = L_-$ is a continuous extension as required. QED

Proof of lemma 5. Each of the arcs $\mathcal{A}_i$ will be considered in turn. The arc $\mathcal{A}_1$ is the subset of the quarter circle $\mathcal{S}$ defined by $\sigma^2 + \tau^2 = c_1$ where $c_1 > 0$ is a small constant. Note that the center of the quarter circle, given by $\sigma = \tau = 0$, is the collinear homothetic solution. The behavior of the solutions $\phi_t(x)$, $x \in \mathcal{A}_1$ will be well approximated by the variational equations along the homothetic orbit, provided c_1 is sufficiently small and only a bounded interval of time is involved. The collinear homothetic orbit has $s = 1, \theta = \sigma = \tau = 0$. The other variables are given by:

$$r(t) = W_0 \operatorname{sech}^2\left(\sqrt{\tfrac{W_0}{2}}\, t\right) \qquad v(t) = -\sqrt{2W_0}\tanh\left(\sqrt{\tfrac{W_0}{2}}\, t\right)$$

where $W_0 = W(1, 0) = \frac{5}{\sqrt{2}}$ is the value of the potential at the central configuration. Note that $v(t)$ decreases monotonically from 0 to $-\sqrt{2W_0}$.

The variational equations along this solution are quite simple. The equations involving the variations in the variables s, σ are decoupled from the others:

$$\begin{aligned} \delta s' &= \delta\sigma \\ \delta\sigma' &= 4W_{ss}\delta s - \tfrac{1}{2}v(t)\delta\sigma. \end{aligned} \tag{17}$$

Moreover, on the homothetic orbit $W_{ss} = -\frac{7}{4\sqrt{2}}$.

Let $x = (r, v, s, \theta, \sigma, \tau) \in \mathcal{A}_1$. Then $s = 1, v = \theta = 0$, $\sigma^2 + \tau^2 = c_1$ and $\sigma + c_2\tau \leq 0$, where $c_2 > 0$ is another constant to be chosen later. It follows that the values of σ are all small and negative. It will now be shown, that for any solution of the variational equations with $\delta s(0) = 0$ and $\delta\sigma(0) < 0$, $\delta\sigma(t)$ reaches 0 after a uniformly bounded time. It follows that if c_1 is sufficiently small, the solution $\phi_t(x)$ reaches a state with $\sigma(t) = 0$ before $\theta(t)$ can reach $\frac{\pi}{3}$ (so the exit is in $\mathcal{B}_2$). Moreover, the value of $v(t)$ will be negative, as is true for the homothetic solution.

Consider a nonzero solution of the linear differential equation (17). The goal is to show that any solution beginning on the negative $\delta\sigma$ axis passes clockwise through the third quadrant in the $(\delta s, \delta\sigma)$-plane and reaches the negative δs axis. To this end, define an angle $\gamma \in [0, \frac{\pi}{2}]$ such that $\tan\gamma = \frac{k\delta s}{\delta\sigma}$ where $k^2 = \frac{7}{\sqrt{2}}$, $k > 0$. To show that this increases from 0 to $\frac{\pi}{2}$ one computes

$$\gamma'(t) = k + \tfrac{1}{2}v(t)\frac{k\delta s\delta\sigma}{k^2\delta s^2 + \delta\sigma^2} \geq k - \tfrac{1}{4}|v(t)| \geq k - \tfrac{1}{4}W_0.$$

It is easy to check that $k - \frac{1}{4}W_0 > 1.55$ and it follows that $\gamma(t)$ reaches $\frac{\pi}{2}$ before time $t = \frac{\pi}{3.1}$ as claimed.

Next, consider $x \in \mathcal{A}_2$. Note that $\mathcal{A}_2$ is contained in the invariant set $\mathcal{I}_3$ of isosceles solutions with m_3 on the axis of symmetry. Some of the points of $\mathcal{A}_2$ may lie in $W_0^s(L_-)$ and therefore never exit $\mathcal{W}$, but lemma 4 provides a continuous extension of the exit map F. Assuming $\phi_t(x)$ does exit $\mathcal{W}$, it must do so through $\mathcal{E} = \mathcal{B}_1 \cup \mathcal{B}_2$ so it suffices to show that exiting through $\mathcal{B}_2$ with $v \geq 0$ is impossible. In other words, if the exit occurs with $\sigma(t) = 0$ then $v(t) < 0$. To prove this, another lemma will be used.

Lemma 6. *There is a function of the form $F = v + f(s)\sigma$ where $f \geq 0$ such that the set $\{F \leq 0\}$ is strictly positively invariant relative to $\mathcal{W} \cap \mathcal{I}_3$, i.e., if $F(x) \leq 0$ for $x \in \mathcal{W} \cap \mathcal{I}_3$ then $F(\phi_t(x)) < 0$ holds for all $t > 0$ as long as the orbit remains in $\mathcal{W} \cap \mathcal{I}_3$.*

This will be proved later on. Note that for any initial condition in $\mathcal{A}_2$, one has $v = 0$ and $\sigma \leq 0$ and so $F \leq 0$. It follows that $F < 0$ at the exit point, i.e., $v < -f\sigma$. If $\sigma = 0$ this implies $v < 0$ as required.

The third arc, $\mathcal{A}_3$, lies entirely in the triple collision manifold $\{r = 0\}$. This is an invariant set, so the equation $r(t) = 0$ continues to hold for all t. In addition, the differential equation for v' on the collision manifold reduces to

$$v' = W(s, \theta) - \tfrac{1}{2}v^2 = \tfrac{1}{2}\frac{\sigma^2 + s^2\tau^2}{(1 + s^2)^2} \geq 0.$$

Initially this is strictly positive and it follows that the exit must occur with $v(t) > 0$ as claimed.

Finally, consider the arc $\mathcal{A}_4$ given by $\sigma = -c_2\tau$. For $c_2 > 0$ small, this is close to the invariant collinear set $\mathcal{C}$ which has been deleted from $\mathcal{W}$. It must be shown that if $x \in \mathcal{A}_4$ then $\phi_t(x)$ exits with $\sigma(t) = 0$ rather than with $\theta(t) = \frac{\pi}{3}$. This will be done by using the variational equations along solutions in $\mathcal{C}$. As before, the equations involving the variations in the variables s, σ are decoupled from the others. This time one finds

$$\begin{aligned} \delta s' &= \delta\sigma \\ \delta\sigma' &= 4W_{ss}\delta s - \tau^2\delta s - \tfrac{1}{2}v(t)\delta\sigma. \end{aligned} \tag{18}$$

but W_{ss} is no longer constant. In fact $W_{ss}(1, \theta) \leq W_{ss}(1, 0) = -\frac{7}{8}\sqrt{2} \approx -1.237$. Furthermore, $W_{ss}(1, \theta) \to -\infty$ as $\theta \to \frac{\pi}{3}$ and this will be the main point of the argument.

Define a different angle $\omega(t)$ in the $(\delta s, \delta\sigma)$-plane such that $\tan\omega = \frac{\delta\sigma}{\delta s}$. Then

$$\omega'(t) = \frac{(-4W_{ss} + \tau^2)\delta s^2 + \delta\sigma^2 + \frac{1}{2}v(t)\delta s\delta\sigma}{\delta s^2 + \delta\sigma^2} \geq \frac{-4W_{ss}\delta s^2 - \sqrt{W/2}\delta s\delta\sigma + \delta\sigma^2}{\delta s^2 + \delta\sigma^2}$$

where the energy bound $v^2 \le 2W(t)$ has been used.

Write the quadratic form in the numerator of $\omega'(t)$ as $A\delta s^2 + 2B\delta s\delta\sigma + C\delta\sigma^2$ where

$$A = -W_{ss}(1,\theta) \qquad B = \sqrt{W/8} \qquad C = 1.$$

Then $A > 0, C > 0$ and $\Delta = AC - B^2 = -W_{ss}(1,\theta) - \frac{1}{8}W(1,\theta)$. It is not too difficult to show that $\Delta > 0$ for $0 \le \theta < \frac{\pi}{3}$. In fact, it is increasing in θ achieving the minimum value $\Delta = \frac{9}{8}\sqrt{2}$ when $\theta = 0$ and $\Delta \to \infty$ as $\theta \to \frac{\pi}{3}$. It follows that the angle $\omega(t)$ is strictly increasing.

For solutions $\phi_t(x)$, $x \in \mathcal{A}_4$, $\theta(t) > 0$ for $t > 0$ and $\theta(t)$ is increasing. Meanwhile the behavior of $s(t), \sigma(t)$ is well-approximated by that of $\delta s(t), \delta\sigma(t)$ obeying (18). It will be shown that the angle $\omega(t)$ reaches $\frac{\pi}{2}$ before $\theta(t)$ reaches $\frac{\pi}{3} - \delta$ where $\delta > 0$ is a small constant. In fact, the change in $\omega(t)$ as $\theta(t)$ varies over $[\frac{\pi}{3} - 2\delta, \frac{\pi}{3} - \delta]$ will be seen to exceed $\frac{\pi}{2}$.

To see this first note that $\theta'(t) = \tau(t) \le \sqrt{2W(1,\theta)}$. Therefore

$$\frac{d\omega}{d\theta} \ge \frac{A\delta s^2 + 2B\delta s\delta\sigma + C\delta\sigma^2}{\sqrt{2W}(\delta s^2 + \delta\sigma^2)}. \tag{19}$$

This has a positive lower bound $\frac{d\omega}{d\theta} \ge d_1 > 0$ for $0 \le \theta < \frac{\pi}{3}$. Therefore, by the time $\theta = 1$, one has $\omega(t) \ge d_1$. This gives a positive lower bound

$$\frac{\delta s^2}{\delta s^2 + \delta\sigma^2} \ge \sin^2 d_1$$

valid for $\theta \ge 1$. Using this in (19) gives

$$\frac{d\omega}{d\theta} \ge \frac{A\sin^2 d_1 - (2B + C)}{\sqrt{2W}} \ge \frac{A\sin^2 d_1}{\sqrt{2W}} - k_1$$

where k_1 is a constant (it turns out that one could even use $k_1 = 1$). Now $A = -4W_{ss}(1,\theta)$ has a pole of order 3 at $\theta = \frac{\pi}{3}$ whereas $W(1,\theta)$ has a simple pole. It follows that there is a constant $k_2 > 0$ such that

$$\frac{d\omega}{d\theta} \ge k_2\delta^{-5/2} - k_1$$

holds for $\theta \in [\frac{\pi}{3} - 2\delta, \frac{\pi}{3} - \delta]$ for δ small enough. Hence the change in $\omega(t)$ as $\theta(t)$ varies over this interval tends to ∞ like $\delta^{-3/2}$ as $\delta \to 0$. If δ is small enough the change in ω will be at least $\frac{\pi}{2}$ as required.

QED

Proof of lemma 6. The lemma refers only to orbits in the isosceles manifold $\mathcal{I}_3$, where $\theta = \tau = 0$. It will be convenient to replace the variable s by a new variable ψ such that

$$s = \frac{\cos\psi}{1 + \sin\psi}.$$

ψ is one of the ordinary spherical coordinates on the shape sphere. In addition, let $\alpha = \psi'$. Then the differential equations can be written

$$\begin{aligned} r' &= vr \\ v' &= W(\psi) - \tfrac{1}{2}v^2 + \tfrac{1}{4}\alpha^2 \\ \psi' &= \alpha \\ \alpha' &= 4\,W_\psi - \tfrac{1}{2}v\alpha. \end{aligned} \tag{20}$$

with energy equation:

$$\frac{1}{2}v^2 + \frac{1}{8}\alpha^2 - W(\psi) = rh \tag{21}$$

and

$$W(\psi) = \frac{1}{\sqrt{1+\cos\psi}} + \frac{2}{\sqrt{1-\frac{1}{2}\cos\psi}}.$$

Using these coordinates, define $F = v - \frac{1}{4}\psi\alpha$. Since σ and α are related by $\sigma = -\frac{\alpha}{1+\sin\psi}$, this can be written in the form $F = v + f\,\sigma$ with $f \geq 0$. It suffices to show that for all $x = (r, v, \psi, \alpha) \in \mathcal{W} \cap \mathcal{I}_3$ such that $F(x) = 0$, the time derivative along the flow satisfies $F'(x) < 0$. Calculating F' from the differential equations and using $F = 0$ to replace v by $\frac{1}{4}\psi\alpha$ gives

$$F'|_{F=0} = \frac{\psi^2}{16}\alpha^2 - W - \psi W_\psi.$$

With $v = \frac{1}{4}\psi\alpha$, the energy equation gives

$$\alpha^2 = \frac{16}{(4+\psi^2)}$$

and so

$$F'|_{F=0} \leq \frac{\psi^2}{4+\psi^2} - W(\psi) - \psi W_\psi.$$

It can be shown that this function is strictly negative for $0 \leq \psi \leq \frac{\pi}{2}$ as required.

QED

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