

# A PROOF OF SAARI'S CONJECTURE FOR THE THREE-BODY PROBLEM IN $\mathbf{R}^d$

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ABSTRACT. The well-known central configurations of the three-body problem give rise to periodic solutions where the bodies rotate rigidly around their center of mass. For these solutions, the moment of inertia of the bodies with respect to the center of mass is clearly constant. Saari conjectured that such rigid motions, called *relative equilibrium solutions*, are the only solutions with constant moment of inertia. This result will be proved here for the Newtonian three-body problem in  $\mathbf{R}^d$  with three positive masses. The proof makes use of some computational algebra and geometry. When  $d \leq 3$ , the rigid motions are the planar, periodic solutions arising from the five central configurations, but for  $d \geq 4$  there are other possibilities.

## 1. INTRODUCTION

It is a well-known property of the Newtonian  $n$ -body problem that the center of mass of the bodies moves along a line with constant velocity. Making a change of coordinates, one may assume that the center of mass is actually constant and remains at the origin. Once this is done, the moment of inertia with respect to the origin provides a natural measure of the size of the configuration. The familiar rigidly rotating periodic solutions of Lagrange provide examples of solutions with constant moment of inertia. Saari conjectured that these are in fact the only such solutions [11]. The goal of this paper is to provide a proof for the three-body problem in  $\mathbf{R}^d$ . This corresponding result for the planar problem was presented in [9].

The three-body problem concerns the motion of three point masses  $m_i > 0$ ,  $i = 1, 2, 3$ , under the influence of their mutual gravitational attraction. Let  $q_i \in \mathbf{R}^d$  denote the positions of the masses and  $\dot{q}_i \in \mathbf{R}^d$  their velocities. Then Newton's equations of motion are

$$(1) \quad m_i \ddot{q}_i = U_{q_i}$$

where

$$U = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$$

is the Newtonian potential energy and  $U_{q_i}$  is the  $d$ -dimensional vector of partial derivatives with respect to the components of  $q_i$ . Here  $r_{ij} = |q_i - q_j|$ ,  $i \neq j$ , are the distances between the masses. The kinetic energy is  $\frac{1}{2}K$  where

$$K = m_1 |\dot{q}_1|^2 + m_2 |\dot{q}_2|^2 + m_3 |\dot{q}_3|^2.$$

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The total energy,

$$(2) \quad H = \frac{1}{2}K - U = h$$

is a constant of motion.

Using the translation symmetry of the problem, one may assume that the center of mass satisfies

$$\frac{1}{m} (m_1 q_1 + m_2 q_2 + m_3 q_3) = 0$$

where  $m = m_1 + m_2 + m_3$  is the total mass. Let

$$I = m_1 |q_1|^2 + m_2 |q_2|^2 + m_3 |q_3|^2$$

be the moment of inertia with respect to the origin. Saari's conjecture is about solutions such that  $I(t)$  is constant, say  $I(t) = c$ . One way to get such a solution is to make the configuration rotate rigidly around the origin. Suppose that  $q_i(t) = R(t)q_i(0)$  where  $R(t) \in \text{SO}(d)$  is a time-dependent rotation in  $\mathbf{R}^d$ . It is known that such a rigid motion is possible only if  $R(t)$  is a rotation with constant angular velocity  $\Omega \in \text{so}(d)$ ; see [12, 1]). Such uniform rigid motions are called *relative equilibrium solutions*.

In dimensions  $d \leq 3$ , the relative equilibrium solutions have been known since the time of Euler and Lagrange [8, 12]. The masses must form a so-called central configuration. The only possible shapes, up to symmetry, are the equilateral triangle and three special collinear configurations, one for each of the rotationally distinct orderings of the masses along the line. The corresponding solutions rotate in a plane containing the masses with constant angular velocity.

However, when  $d = 4$ , Albouy and Chenciner showed that there are other relative equilibrium solutions. For example, when the masses are equal, the shape can be an arbitrary isosceles triangle [1]. The corresponding rigid motions involve simultaneous rotations around two orthogonal planes in  $\mathbf{R}^4$ . If the rotation rates in the two planes are incommensurable, the resulting solution will be quasi-periodic. Note that for the three-body problem, the initial position and velocity vectors of any solution with center of mass at the origin will always span a subspace of dimension  $d \leq 4$  and the solution will then remain in this subspace for all time. Thus there is no point in considering  $d > 4$ .

A priori, rigid rotation is a stronger condition than constant moment of inertia, but the goal of this paper is to prove:

**Theorem 1.** *A solution of the three-body problem has constant moment of inertia if and only if it is a relative equilibrium solution.*

Here is an outline of the proof. Consider a solution with constant moment of inertia. To show that it is a rigid motion, it suffices to show that the mutual distances  $r_{ij}$  are constant. If not, then they would take on infinitely many different values. Using the assumption of constant moment of inertia, one can derive a set of algebraic equations involving the variable  $r_{ij}$  and certain velocity variables. The proof will consist of showing that among the solutions of these equations, only finitely many distinct values of the  $r_{ij}$  are attained.

The algebraic equations are derived from the equations of motion and the assumption of constant moment of inertia:  $I(t) = c$ . Differentiating this equation

gives:

$$\begin{aligned}\frac{1}{2}\dot{I} &= m_1q_1 \cdot \dot{q}_1 + m_2q_2 \cdot \dot{q}_2 + m_3q_3 \cdot \dot{q}_3 = 0 \\ \frac{1}{2}\ddot{I} &= m_1|\dot{q}_1|^2 + m_2|\dot{q}_2|^2 + m_3|\dot{q}_3|^2 + q_1 \cdot U_{q_1} + q_2 \cdot U_{q_2} + q_3 \cdot U_{q_3} \\ &= K - U = 0\end{aligned}$$

where the second equation uses Newton's law of motion and the fact that  $U$  is homogeneous of degree  $-1$ . The equation  $K = U$  together with the energy equation (2) shows that  $K = U = -2h$ . Now any bounded solution of the three-body problem has energy  $h < 0$  and by rescaling, one may assume without loss of generality that  $h = -\frac{1}{2}$  and so  $K = U = 1$ . The constancy of  $U$  also implies that the time-derivatives of  $U$  of all orders must vanish along the solution. Each of these provides a constraint on the positions and velocities. Finally, for solutions in  $\mathbf{R}^4$  (which include all lower-dimensional solutions as special cases) there are two rotation-invariant angular momentum integrals  $\mathbf{C}_0, \mathbf{C}_1$  whose values can be fixed.

Therefore, for any solution with constant moment of inertia, the following ten equations must hold:

$$(3) \quad \begin{array}{cccccc} I = c & U = 1 & K = 1 & C_0 = \omega_0 & C_1 = \omega_1 \\ \dot{I} = 0 & \dot{U} = 0 & \dot{K} = 0 & \dot{C}_0 = 0 & \dot{C}_1 = 0 \end{array}$$

The reason for using ten equations is that in section 3 these equations will be expressed in terms of ten variables, among which are the three mutual distances  $r_{ij}$ . After some elimination, a set of five equations for the  $r_{ij}$  and two other variables will be obtained. Then in section 4 it will be shown that solutions of these equations admit only finitely many values of  $r_{ij}$ , completing the proof of the theorem.

The proof presented here would not be feasible without the use of computers. Mathematica was used throughout to perform symbolic computations on the very large expressions which arise [13]. Porta was used to compute convex hulls [4]. A Mathematica notebook presenting details of all computations with extensive comments can be found at [10].

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## 2. LAGRANGE'S EQUATIONS OF MOTION

Equations of motion for the distances  $r_{ij}$  were first derived by Lagrange [8]. Albouy and Chenciner generalized these equations to the  $n$ -body problem [1, 2]. The presentation here follows that in [2] with a few changes of notation.

Let  $q_{ij} = q_i - q_j \in \mathbf{R}^d$ . Then Newton's equations (1) give:

$$\ddot{q}_{ij} = -\Sigma_{ij}q_{ij} - \frac{1}{2} \sum_{k \neq i,j} m_k (q_{ki} + q_{kj}) \left( \frac{1}{r_{ik}^3} - \frac{1}{r_{jk}^3} \right) \quad 1 \leq i \neq j \leq 3$$

where

$$\Sigma_{ij} = \frac{m_i + m_j}{r_{ij}^3} + \frac{1}{2} \sum_{k \neq i,j} m_k \left( \frac{1}{r_{ik}^3} + \frac{1}{r_{jk}^3} \right).$$

Using the variables  $q_{ij}$  eliminates the translation symmetry of the three-body problem. To eliminate the rotation symmetry, Lagrange introduced the following variables:

$$s_{ij} = |q_{ij}|^2 = r_{ij}^2 \quad s'_{ij} = q_{ij} \cdot \dot{q}_{ij} \quad s''_{ij} = |\dot{q}_{ij}|^2$$

and

$$\rho = q_{13} \cdot \dot{q}_{23} - q_{23} \cdot \dot{q}_{13}.$$

Clearly  $s_{ij} = s_{ji}$  so only three of these variables are needed. To be definite, the variables with subscripts 12, 31, 23 will be adopted. After imposing the same convention for the  $s'_{ij}$  and  $s''_{ij}$ , one has ten variables in all.

Lagrange derived differential equations for these variables which may be found in [2]. However, to avoid square roots, it is convenient to replace  $s_{ij}$  by  $r_{ij}$  and  $s'_{ij}$  by  $v_{ij} = s'_{ij}/r_{ij} = \dot{r}_{ij}$ . Then the differential equations are:

$$\begin{aligned} \dot{r}_{ij} &= v_{ij} \\ \dot{v}_{ij} &= \frac{s''_{ij} - v_{ij}^2}{r_{ij}} - r_{ij} \Sigma_{ij} - \frac{1}{2} m_k (r_{ki}^2 - r_{jk}^2) \left( \frac{1}{r_{ki}^3} - \frac{1}{r_{jk}^3} \right) \\ (4) \quad \dot{s}''_{ij} &= -2r_{ij} v_{ij} \Sigma_{ij} - m_k (r_{ki} v_{ki} - r_{jk} v_{jk} - \rho) \left( \frac{1}{r_{ki}^3} - \frac{1}{r_{jk}^3} \right) \\ \dot{\rho} &= \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ m_1(r_{23}^2 - r_{31}^2 - r_{12}^2) & m_2(r_{31}^2 - r_{12}^2 - r_{23}^2) & m_3(r_{12}^2 - r_{23}^2 - r_{31}^2) \\ r_{23}^{-3} & r_{31}^{-3} & r_{12}^{-3} \end{vmatrix}. \end{aligned}$$

Here  $(i, j, k)$  is always a cyclic permutation of  $(1, 2, 3)$ . By construction the ten variables  $r_{ij}, v_{ij}, s''_{ij}, \rho$  are all invariant under rotations and translations in  $\mathbf{R}^d$ . The differential equations (4) give a remarkable dimension-independent reduction of the three-body problem.

A *relative equilibrium solution* of the three-body problem will be defined as one whose Lagrange variables give an equilibrium point of the reduced differential equations (4). From the first differential equation, it is clear that for any such solution, the mutual distances are constant, i.e., it will be a rigid motion. Less obvious is the fact that the only rigid motions which solve (4) come from its equilibrium points [1]. Thus rigid motion solutions and relative equilibrium solutions coincide.

The moment of inertia and the various integrals of the problem can all be expressed in terms of Lagrange's variables. First one has

$$\begin{aligned} I &= \frac{1}{m} (m_1 m_2 r_{12}^2 + m_1 m_3 r_{31}^2 + m_2 m_3 r_{23}^2) \\ (5) \quad U &= \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{31}} + \frac{m_2 m_3}{r_{23}} \\ K &= \frac{1}{m} (m_1 m_2 s''_{12} + m_1 m_3 s''_{31} + m_2 m_3 s''_{23}) \end{aligned}$$

where, as before,  $m = m_1 + m_2 + m_3$ .

The angular momentum is more complicated, especially in dimension  $d = 4$ . Recall that in  $\mathbf{R}^3$  the angular momentum vector is not invariant under rotations. However its length is invariant and so provides an integral of the reduced equations of motion. In  $\mathbf{R}^4$ , there are two independent rotation-invariant integrals. The method used to derive them here follows [1, p.161–162]. Let  $\beta$  denote the symmetric

$3 \times 3$  matrix:

$$\beta = \begin{bmatrix} 0 & -\frac{1}{2}s_{12} & -\frac{1}{2}s_{31} \\ -\frac{1}{2}s_{12} & 0 & -\frac{1}{2}s_{23} \\ -\frac{1}{2}s_{31} & -\frac{1}{2}s_{23} & 0 \end{bmatrix}$$

and let  $\gamma$  and  $\delta$  be defined in a similar way but with entries involving  $s'_{ij}$  and  $s''_{ij}$  respectively. Also, let  $\hat{\rho}$  be the antisymmetric  $3 \times 3$  matrix:

$$\hat{\rho} = \begin{bmatrix} 0 & -\rho_{12} & \rho_{31} \\ \rho_{12} & 0 & -\rho_{23} \\ -\rho_{31} & \rho_{23} & 0 \end{bmatrix}$$

where  $\rho_{ij} = \frac{1}{2}(q_i \cdot \dot{q}_j - q_j \cdot \dot{q}_i)$ . The  $6 \times 6$  block matrix

$$\mathcal{E} = \begin{bmatrix} \beta & \gamma - \hat{\rho} \\ \gamma + \hat{\rho} & \delta \end{bmatrix}$$

is used in [1] to represent the reduced state of the three-body problem. It is a tensor version of Lagrange's variables, where the last Lagrange variable is given by  $\rho = 2(\rho_{12} + \rho_{31} + \rho_{23})$ .

Now let  $\omega_\mu$  denote the  $6 \times 6$  block matrix

$$\omega_\mu = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix} \quad \mu = \frac{1}{m} \begin{bmatrix} m_1(m_2 + m_3) & -m_1m_2 & -m_1m_3 \\ -m_1m_2 & m_2(m_1 + m_3) & -m_2m_3 \\ -m_1m_3 & -m_2m_3 & m_3(m_1 + m_2) \end{bmatrix}.$$

Consider the linear map  $\omega_\mu \mathcal{E} : \mathbf{R}^6 \rightarrow \mathbf{R}^6$ . Let  $P$  be the plane in  $\mathbf{R}^3$  with equation  $\xi_1 + \xi_2 + \xi_3 = 0$  ( $P$  is called  $\mathcal{D}^*$  in [1]). If one identifies  $\mathbf{R}^6$  with  $\mathbf{R}^3 \times \mathbf{R}^3$  then  $\omega_\mu \mathcal{E}$  maps the four-dimensional subspace  $P \times P$  into itself. The characteristic polynomial of  $\omega_\mu \mathcal{E}|_{P \times P}$  is of the form  $\lambda^4 + C_1 \lambda^2 + C_0$  where  $C_i$  are polynomials in the variables  $s_{ij}, s'_{ij}, s''_{ij}, \rho_{ij}$  but the  $\rho_{ij}$  only appear in the combination  $\rho_{12} + \rho_{31} + \rho_{23}$  so can be replaced by  $\rho$ . These polynomials are the rotation-invariant angular momenta. To compute them, a matrix of  $\omega_\mu \mathcal{E}|_{P \times P}$  was calculated with respect to the basis  $\{v_1, 0\}, \{v_2, 0\}, \{0, v_1\}, \{0, v_2\}$  for  $P \times P$  where  $v_1 = (1, -1, 0), v_2 = (1, 1, -2)$ .

The integral  $C_1$  can be written in a fairly simple form (compare [2]):

$$(6) \quad C_1 = \frac{m_1 m_2 m_3}{2m} (\phi - \psi + \rho^2) + IK - J^2$$

where  $I, K$  are given by (5)

$$(7) \quad \begin{aligned} \phi &= -s'_{12}{}^2 - s'_{31}{}^2 - s'_{23}{}^2 + 2s_{31}s_{23} + 2s_{12}s_{23} + 2s_{12}s_{31} \\ \psi &= -s_{12}s''_{12} - s_{31}s''_{31} - s_{23}s''_{23} \\ &\quad + s_{31}s''_{23} + s_{12}s''_{23} + s_{12}s''_{31} + s''_{31}s_{23} + s''_{12}s_{23} + s''_{12}s_{31} \\ J &= \frac{1}{m}(m_1 m_2 s'_{12} + m_1 m_3 s'_{31} + m_2 m_3 s'_{23}) \end{aligned}$$

The formula for  $C_0$  is more complicated. It can be written:

$$(8) \quad C_0 = \frac{m_1^2 m_2^2 m_3^2}{16m^2} [(\phi - \psi + \rho^2)^2 + 8D\rho - 4P]$$

where  $D$  is the determinant:

$$D = \begin{vmatrix} s_{12} & s_{31} & s_{23} \\ s'_{12} & s'_{31} & s'_{23} \\ s''_{12} & s''_{31} & s''_{23} \end{vmatrix}$$

and

$$\begin{aligned}
P = & 2(s_{23}s''_{31} + s_{31}s''_{23})s'_{12}{}^2 - 2(s_{12}\sigma''_{12} + s''_{12}\sigma_{12})s'_{23}s'_{31} \\
& + 2(s_{23}s''_{12} + s_{12}s''_{23})s'_{31}{}^2 - 2(s_{31}\sigma''_{31} + s''_{31}\sigma_{31})s'_{12}s'_{23} \\
& + 2(s_{12}s''_{31} + s_{31}s''_{12})s'_{23}{}^2 - 2(s_{23}\sigma''_{23} + s''_{23}\sigma_{23})s'_{12}s'_{31} \\
& + s_{12}^2s''_{31}s''_{23} - s''_{12}\sigma''_{12}s_{23}s_{31} \\
& + s_{31}^2s''_{12}s''_{23} - s''_{31}\sigma''_{31}s_{12}s_{23} \\
& + s_{23}^2s''_{31}s''_{12} - s''_{23}\sigma''_{23}s_{12}s_{31}.
\end{aligned}$$

where

$$\sigma_{ij} = (s_{jk} + s_{ki} - s_{ij}) \quad \sigma''_{ij} = (s''_{jk} + s''_{ki} - s''_{ij}).$$

One only needs to make the substitutions  $s_{ij} = r_{ij}^2$  and  $s'_{ij} = r_{ij}v_{ij}$  into these formulas to obtain the angular momentum integrals in the form  $C_i(r_{ij}, v_{ij}, s''_{ij}, \rho)$ .

### 3. CONSTANT MOMENT OF INERTIA SOLUTIONS

The goal of this section is to derive algebraic constraints on the coordinates of solutions with constant moment of inertia. Actually, the final equations will only apply to solutions which are hypothetical counterexamples to theorem 1. In addition to having constant moment of inertia, such a solution would have nonconstant mutual distances. Therefore the velocity variables  $v_{ij} = \dot{r}_{ij}$  are not all zero, except at isolated values of the time. As time varies, such a solution would sweep out a curve containing infinitely many points with different values of the mutual distances  $r_{ij}$  and such that the  $v_{ij}$  are not all zero. By deriving equations which apply to such points and then showing that they do not have an infinity of such solutions, the theorem will be proved.

The first step is just to express the ten equations (3) in Lagrange's variables. The first three equations

$$I = c \quad U = 1 \quad K = 1$$

are straightforward. From (5), one gets equations involving rational functions whose denominators are monomials in the nonzero quantities  $r_{ij}$  and  $m$ . Clearing the denominators gives polynomial equations  $g_i = 0$ ,  $1 \leq i \leq 3$ . One could take a similar approach to the angular momentum equations to get two further polynomials  $g_4, g_5$ , but as will be seen shortly, a significant simplification of these formulas is possible for solutions with constant moment of inertia.

Using the differential equations (4) together with the chain rule one can find the time derivative of any function  $f(r_{ij}, v_{ij}, s''_{ij}, \rho)$ , i.e., the derivative in the direction of the vectorfield. For example

$$\begin{aligned}
\dot{I} &= \frac{2}{m}(m_1m_2r_{12}v_{12} + m_1m_3r_{31}v_{31} + m_2m_3r_{23}v_{23}) = 0 \\
\dot{U} &= -\frac{m_1m_2v_{12}}{r_{12}^2} + \frac{m_1m_3v_{31}}{r_{31}^2} + \frac{m_2m_3v_{23}}{r_{23}^2} = 0
\end{aligned}
\tag{9}$$

Note that  $\dot{I} = 2J$  where  $J$  is given by (7) together with the substitutions for  $s'_{ij}$ . Thus  $J = 0$  for constant moment of inertia solutions.

Of course the expressions for the higher time derivatives of  $U$  are more complicated, but they are all rational functions whose denominators are monomials in the

$r_{ij}$  and  $m$ . Clearing denominators in the equations

$$\dot{I} = 0 \quad \dot{U} = 0 \quad \ddot{U} = 0 \quad \ddot{U} = 0 \quad \ddot{U} = 0$$

gives five more polynomial equations  $g_i = 0$ ,  $6 \leq i \leq 10$ .

From (6) and the equations above one finds:

$$C_1 = \frac{m_1 m_2 m_3}{2m} (\phi - \psi + \rho^2) + c = \omega_1.$$

Clearing denominators gives a polynomial equation  $g_5 = 0$ . This shows that the quantity  $(\phi - \psi + \rho^2)$  is constant and this can be used to simplify  $C_0$ . Instead of setting  $C_0 = \omega_0$  one can set

$$2D\rho - P = \omega_0$$

where  $\omega_0$  is a different constant. This gives the last equation  $g_4 = 0$ .

Next, the variables  $v_{ij}$  and  $s''_{ij}$  will be eliminated. It is easy to see that equations (9) uniquely determine the direction of the three-dimensional vector  $v = (v_{12}, v_{31}, v_{23})$  unless the three mutual distances  $r_{ij}$  are all equal. Namely the ‘‘gradient vectors’’  $\nabla I = (I_{r_{12}}, I_{r_{31}}, I_{r_{23}})$  and  $\nabla U = (U_{r_{12}}, U_{r_{31}}, U_{r_{23}})$  are linearly independent in this case. If the distances are equal, their values are uniquely determined by, for example, the equation  $U = 1$ . Since the goal is to show that only finitely many values of the mutual distances are possible, one may assume a priori that the distances are not equal. In other words, if assuming that the  $r_{ij}$  are not equal leads to equations which restrict the  $r_{ij}$  to a finite set, then the original equations also restrict  $r_{ij}$  to a finite set.

Let  $V(r_{ij})$  denote some nonzero scalar multiple the cross-product  $\nabla I \times \nabla U$ . Then one can replace the three variables  $v_{ij}$  by a single variable  $d$  such that  $v = dV$ . After this is done, equations  $g_6 = g_7 = 0$  will hold automatically so one has eight remaining equations for the eight unknowns  $r_{ij}, s''_{ij}, d, \rho$ .

Next, it turns out that the formulas for  $g_3, g_8, g_9$  are linear in  $s''_{ij}$  (these are the equations  $K = 1, \ddot{U} = \ddot{U} = 0$ ). Apart from nonzero factors of powers of the variables  $m_i, r_{ij}$  and a factor of  $d$ , the determinant of the  $3 \times 3$  coefficient matrix is

$$m_1 r_{23} (r_{12}^3 - r_{31}^2)^2 + m_2 r_{31} (r_{12}^3 - r_{23}^2)^2 + m_3 r_{12} (r_{31}^3 - r_{23}^2)^2$$

which vanishes only when the mutual distances are all equal.

From this point on it will be assumed that  $d \neq 0$ . As noted above, it is only necessary to derive equation applicable when the velocity variables  $v_{ij} = \dot{r}_{ij}$  are not all zero, which implies  $d \neq 0$ . Assuming  $d \neq 0$  and that the distances are not equal, one can solve the equations  $g_3 = g_8 = g_9 = 0$  for the  $s''_{ij}$ . Substituting into the remaining equations  $g_1, g_2, g_4, g_5, g_{10}$  gives a system of five polynomial equations for the five variables  $r_{ij}, d, \rho$ . The constants  $m_i, c, \omega_0, \omega_1$  appear as parameters. The first two equations (derived from  $I = c$  and  $U = 1$ ) are quite simple

$$(10) \quad \begin{aligned} g_1 &= m_1 m_2 r_{12}^2 + m_1 m_3 r_{31}^2 + m_2 m_3 r_{23}^2 - mc = 0 \\ g_2 &= m_2 m_3 r_{12} r_{31} + m_1 m_3 r_{12} r_{23} + m_1 m_2 r_{31} r_{23} - r_{12} r_{31} r_{23} = 0. \end{aligned}$$

The other three are too complicated to include here (see [10]). A few terms of each are shown here:

$$\begin{aligned}
(11) \quad g_4 &= -2m_1^2m_2^6m_3^2r_{12}^{22}r_{31}^6 - 18d\rho m_1^2m_2^6m_3^2r_{12}^{21}r_{31}^9r_{23}^3 \\
&\quad - 5d^2m_1^2m_2^7m_3^2r_{12}^2r_{31}^{10}r_{23}^3 + 25d^4m_1^2m_2^8m_3^2r_{12}^{30}r_{31}^{14}r_{23}^6 \\
&\quad - 54d^3\rho m_1^3m_2^6m_3^2r_{12}^{27}r_{31}^{10}r_{23}^7 + \dots = 0 \\
g_5 &= 2m_1m_2^3m_3r_{12}^{13}r_{31}^4 + 3\rho^2m_1m_2^2m_3r_{12}^9r_{31}^4r_{23}^3 \\
&\quad - 5d^2m_1m_2^4m_3r_{12}^{17}r_{31}^8r_{23}^3 + \dots = 0 \\
g_{10} &= -4m_1^2m_2^3m_3^3r_{12}^{10}r_{31}^2 - 9d\rho m_1^2m_2^4m_3^2r_{12}^{13}r_{31}^3r_{23} \\
&\quad - 5d^2m_1^2m_2^5m_3^2r_{12}^{18}r_{31}^4r_{23} + 20d^4m_1^2m_2^6m_3^2r_{12}^{22}r_{31}^8r_{23}^4 + \dots = 0.
\end{aligned}$$

After collecting terms with the same powers of the five variables  $r_{ij}, d, \rho$ , the total number terms is 2238, 244 and 509 respectively.

If a counterexample to theorem 1 exists, then the equations constructed in this section would admit solutions with infinitely many distinct values of the mutual distances  $r_{ij}$ . However, in the next section it will be shown that if the parameters satisfy  $m_i > 0, c \neq 0$ , this is not the case.

#### 4. FINITENESS PROOF

It remains to show that among all solutions  $r_{ij}, d, \rho$  of the five equations  $g_1 = g_2 = g_4 = g_5 = g_{10} = 0$ , only finitely many values of the mutual distances  $r_{ij}$  are attained. This will be proved even allowing all the variables to be complex but with the assumptions  $r_{ij} \neq 0, d \neq 0$ . The method is a variation on the one used for the planar problem in [9] and further developed in [5]. It is based on fundamental work of Bernstein, Khovanskii and Kushnirenko [3, 6, 7]. which is sometimes described as BKK theory.

**4.1. BKK Theory.** Assume for the sake of contradiction that there exists infinitely many values of the mutual distances among the solutions. Then, as in [9, 5], the algebraic variety  $V \subset \mathbf{C}^{*4} \times \mathbf{C}$  determined by the five polynomials contains a subvariety which projects dominantly onto one of the copies of  $\mathbf{C}^* = \mathbf{C} \setminus 0$  corresponding to the variables  $r_{ij}$  (projections of complex varieties onto coordinate axes are either finite sets or else the complements of finite sets; the latter case is called *dominant*). It follows that the polynomial equations admit solutions in the form of Puiseux series in some variable,  $t$  (which is unrelated to the time variable of the equations of motion). In other words, there would be fractional power series  $r_{ij}(t), d(t), \rho(t)$  which solve the equations identically in  $t$ . Moreover, these series  $r_{ij}(t), d(t)$  would not be identically 0 and at least one of the series  $r_{ij}(t)$  would be nonconstant (in fact one can even use one of these variables as the Puiseux parameter,  $t$ ). To get the contradiction, one only needs to show that such series solutions are impossible.

Here are slight variations of some results from [5, sec.3] which make the form of the series more explicit. Consider the general problem of solving  $m$  polynomial equations in  $n$  unknowns:

$$(12) \quad f_i(x_1, \dots, x_n) = \sum_k c_k x_1^{k_1} \dots x_n^{k_n} = 0 \quad i = 1, \dots, m$$

where the exponent vector  $k = (k_1, \dots, k_n)$  runs over a finite set  $S_i \subset \mathbf{Z}^n$  (called the support of  $f_i$ ). Suppose that one wants complex solutions where the first  $k$  variables are to be nonzero, i.e., one wants  $x = (x_1, \dots, x_n) \in \mathbf{T}$  where  $\mathbf{T} = \mathbf{C}^{*k} \times \mathbf{C}^{k'}$ . The result in [5] assumes that  $k = n, k' = 0$  but the proof is virtually the same.

**Proposition 1.** *Suppose that a system of  $m$  polynomial equations  $f_i(x) = 0$  in  $n$  variables defines an infinite variety  $V \subset \mathbf{T}$ . Then there is a nonzero rational vector  $\alpha = (\alpha_1, \dots, \alpha_n)$ , a point  $a = (a_1, \dots, a_n) \in \mathbf{T}$ , and Puiseux series  $x_j(t) = a_j t^{\alpha_j} + \dots$ ,  $j = 1, \dots, n$ , convergent in some punctured neighborhood  $U$  of  $t = 0$ , such that  $f_i(x_1(t), \dots, x_n(t)) = 0$  in  $U$ ,  $i = 1, \dots, m$ . Moreover, if the projection from  $V$  onto the  $x_1$ -axis is dominant, there exists such a series solution with  $x_1(t) = t$  and another with  $x_1(t) = t^{-1}$ .*

The vector  $\alpha$  of leading exponents will be called the *order vector* of the Puiseux solution. Note however, that some of the  $a_i$  with  $i > k$  could be zero. In this case  $\alpha_i$  will not really be the lowest-order exponent of the series  $x_i(t)$ . Of course, if a series is not identically zero it will have some leading term and one could insist that  $\alpha_i$  be taken as the leading exponent, but for  $i > k$  the possibility of an identically zero series is not excluded,

To apply this test one needs a way to show that Puiseux solutions of a given order do not exist. Bernstein provides a simple test based on the leading terms of the solutions. Substituting  $x_j(t)$  into equations  $f_i(x) = 0$  and reading off the coefficients of the lowest order terms in  $t$  gives a *reduced system*

$$(13) \quad f_{i\alpha}(a_1, \dots, a_n) = \sum_{\alpha \cdot k = \mu_i} c_k a_1^{k_1} \dots a_n^{k_n} = 0 \quad i = 1, \dots, m$$

where  $\mu_i = \min_{l \in S_i} \alpha \cdot l$ . The equation  $\alpha \cdot k = \mu_i$  which determines which terms of  $f_i$  appear in the reduced equation has a beautiful geometrical interpretation. Let  $P_i$  be the *Newton polytope* of  $f_i$ , i.e., the convex hull of the support  $S_i$ . Then  $\alpha \cdot k = \mu_i$  defines a supporting hyperplane of  $P_i$  for which  $\alpha$  is an inward normal vector. The hyperplane defines a face of the Newton polytope and the exponent vectors  $k$  which appear in the reduced equation are the vertices of  $P_i$  which lie on this face.

Since the coefficients of the leading terms of any series solution must vanish, one has a simple test for nonexistence of a solution of order  $\alpha$ :

**Proposition 2.** *Let  $\alpha$  be a nonzero rational vector. If the reduced system (13) has no solutions in  $\mathbf{T}$  then there does not exist a Puiseux series solution of the full system  $f_i(x) = 0$  of order  $\alpha$ .*

As Bernstein noted, there are only finitely many possible reduced systems arising from a given system of equations. To find them it suffices to construct the Minkowski sum polytope  $P = P_1 + \dots + P_m$  of the  $m$  Newton polytopes, then for each face of  $P$  choose an inward pointing normal vector  $\alpha$ . If the resulting reduced system has no solutions in  $\mathbf{T}$  then all of the inward normals for that face of  $P$  are eliminated.

These ideas will be applied to the system of five equations  $g_1 = g_2 = g_4 = g_5 = g_{10} = 0$ . The Newton polytopes of these polynomials are the convex hulls of the supports which contain 4, 4, 2238, 244, 509 points respectively. These polytopes live in  $\mathbf{R}^5$  because there are five variables. However, the equations have a special structure which makes it possible to work with only three variables at a time.

- |                                      |   |
|--------------------------------------|---|
| 1. $k_{12} + k_{31} \geq 1$          | 8. $k_{12} \geq 0$                      |
| 2. $k_{12} + k_{23} \geq 1$          | 9. $k_{31} \geq 0$                      |
| 3. $k_{31} + k_{23} \geq 1$          | 10. $k_{23} \geq 0$                     |
| 4. $-k_{12} - k_{31} \geq -4$        | 11. $-k_{12} \geq -3$                   |
| 5. $-k_{12} - k_{23} \geq -4$        | 12. $-k_{31} \geq -3$                   |
| 6. $-k_{31} - k_{23} \geq -4$        | 13. $-k_{23} \geq -3$                   |
| 7. $k_{12} + k_{31} + k_{23} \geq 2$ | 14. $-k_{12} - k_{31} - k_{23} \geq -5$ |

TABLE 1. Facets for the Minkowski sum  $P_1 + P_2$ .

Namely, the first two equations (10) involve only the three mutual distances. These two equations restrict the  $r_{ij}$  to an algebraic curve in  $\mathbf{C}^{*3}$ . By analyzing the three-dimensional Newton polytopes of  $g_1, g_2$ , it will be shown in section 4.2 that there are essentially only two possibilities for the Puiseux series  $r_{ij}(t)$ . Substitution of these series into  $g_4, g_5, g_{10}$  gives three equations in the variables  $d, \rho, t$ . In section 4.3 elementary methods are used to show that these equations have no series solutions  $d(t), \rho(t)$ .

**4.2. Puiseux Expansions for the Distances.** In this subsection, it will be shown that the equations (10) admit only two types of Puiseux expansions  $r_{ij}(t)$  for the mutual distances. By proposition 2, the order vector  $\alpha = (\alpha_{12}, \alpha_{31}, \alpha_{23})$  of the series must be such that the corresponding reduced system has nonzero solutions  $(a_{12}, a_{31}, a_{23}) \in \mathbf{T} = \mathbf{C}^{*3}$ . The Newton polytopes  $P_1, P_2$  of  $g_1, g_2$  are the convex hulls in  $\mathbf{R}^3$  of the vectors of exponents  $k = (k_{12}, k_{31}, k_{23})$  of the monomials which appear. Since each polynomial contains 4 monomials, the individual Newton polytopes are simple tetrahedra. As candidates for the order vectors,  $\alpha$ , one needs to check the inward normals to the faces of the Minkowski sum polytope  $P_1 + P_2$ . Using Porta [4], one finds that their Minkowski sum is a polytope with 12 vertices and 14 facets. The inequalities defining the facets are shown in table 1

For each inequality, the coefficients of the left-hand side give an inward normal for the corresponding facet. For example,  $\alpha = (1, 0, 0)$  is a normal for facet 8. This represents a possible order-vector of a Puiseux series solution which would therefore look like:

$$r_{12}(t) = a_{12}t + \dots \quad r_{31}(t) = a_{31} + \dots \quad r_{23}(t) = a_{23} + \dots$$

with all leading coefficients nonzero. The leading coefficients would have to satisfy the reduced system of equations corresponding to  $\alpha$ . These are:

$$\begin{aligned} m_1 m_3 a_{13}^2 + m_2 m_3 a_{23}^2 - mc &= 0 \\ m_1 m_2 a_{31} a_{23} &= 0 \end{aligned}$$

The second reduced equation consists of a single term in the nonzero variables  $a_{ij}$ . In this case the reduced system is solvable only if the coefficient  $m_1 m_2$  vanishes, which violates the assumption on the masses. Therefore the reduced systems has no nonzero solutions and it follows that there does not exist a Puiseux solution with order vector  $\alpha$ . A facet such that at least one of the reduced equations consists of a single term will be called *trivial*. It turns out that facets 7–14 are all trivial.

This leaves facets 1–6. By proposition 1 one can choose to look for Puiseux solutions with one of the  $r_{ij} = t$ . The corresponding face of the Minkowski sum

polytope would have to have an inward normal  $\alpha$  with a 1 in the  $\alpha_{ij}$  position (or, allowing for a rescaling the normal vector, at least a positive entry in that position). This rules out facets 4–6. Finally, facets 1–3 are the same up to permutation symmetry so it suffices to consider one of them.

In searching for possible order-vectors, one should also consider inward normal vectors to lower-dimensional faces of  $P_1 + P_2$ . For three-dimensional polytopes, this means the edges and vertices. Reduced systems arising from vertices necessarily involve just one term from each of the equations and so are automatically trivial. It turns out the reduced equations for the edges are also trivial. Thus, up to symmetry, the only possible Puiseux series for  $r_{ij}(t)$  arise from facet 1.

The order vector for such a series is  $\alpha = (1, 1, 0)$ . One of the first two variables can be set equal to the Puiseux parameter  $t$ . Make the substitution

$$r_{12} = t \quad r_{31} = t u_{31} \quad r_{23}(t) = u_{23} + \dots$$

where  $u_{31}(t) = a_{31} + \dots$ ,  $u_{23}(t) = a_{23} + \dots$  and the leading coefficients are to be nonzero. Then (10) gives (after cancelling a factor of  $t$  from the second equation):

$$(14) \quad \begin{aligned} G_1(u_{31}, u_{23}, t) &= m_2 m_3 u_{23}^2 - mc + t^2 m_1 (m_2 - m_3 u_{31}^2) = 0 \\ G_2(u_{31}, u_{23}, t) &= m_1 u_{23} (m_3 + m_2 u_{31}) + t (m_2 m_3 - u_{23}) = 0. \end{aligned}$$

At  $t = 0$  one obtains the reduced system for the leading coefficients:

$$(15) \quad \begin{aligned} m_2 m_3 a_{23}^2 - mc &= 0 \\ m_1 a_{23} (m_3 + m_2 a_{31}) &= 0. \end{aligned}$$

Thus the leading coefficients are given by:

$$(16) \quad a_{31} = -\frac{m_3}{m_2} \quad a_{23}^2 = \frac{mc}{m_2 m_3}.$$

Moreover, the Jacobian matrix of  $(G_1, G_2)$  with respect to  $(u_{31}, u_{23})$  evaluated at  $(a_{31}, a_{23})$  and  $t = 0$  is  $-2m_1 m_2^2 m_3 a_{23}^2 \neq 0$ . It follows from the implicit function theorem that  $u_{31}(t)$  and  $u_{23}(t)$  are actually power series in  $t$ , not just Puiseux series.

The form of  $G_1$  implies that the rest of the series  $u_{23}(t)$  begins at order at least 2. Then setting

$$u_{31}(t) = a_{31} + b_{31} t + \dots \quad u_{23}(t) = a_{23} + b_{23} t^2 + \dots$$

in (14), where  $a_{31}, a_{23}$  satisfy (16), one finds:

$$(17) \quad b_{31} = \frac{m_3(m_2 m_3 - a_{23})}{m_1 m_2^2 a_{23}} \quad b_{23} = -\frac{m_1(m_2^3 + m_3^3)}{2m_2^2 m_3 a_{23}}.$$

Under the standing assumption of positive masses,  $b_{23} \neq 0$  but it is possible that  $b_{31} = 0$ . In this case, it will be necessary to look for the next nonzero term in the series.

To this end, suppose that, in fact,  $b_{31} = 0$ , that is,  $a_{23} = m_2 m_3$ . Then (16) determines the value of the moment of inertia:  $c = \frac{m_2^3 m_3^3}{m}$ . Moreover, one finds that the next nonzero term in  $u_{31}(t)$  must be of order at least 3. Setting

$$u_{31}(t) = a_{31} + \hat{b}_{31} t^3 + \dots \quad u_{23}(t) = a_{23} + b_{23} t^2 + \dots$$

in (14) and using (16), one finds:

$$(18) \quad \hat{b}_{31} = -\frac{m_1(m_2^3 + m_3^3)}{2m_1 m_2^2 m_3^2} \quad b_{23} = -\frac{m_1(m_2^3 + m_3^3)}{2m_2^3 m_3^2}.$$

Note that the value of  $b_{23}$  is the same as before, given the assumed value of  $a_{23}$ . Since  $\hat{b}_{31} \neq 0$ , this will be the next nonzero term in this case. For reference, the values of  $c$  and  $a_{ij}$  in this case are

$$(19) \quad a_{31} = -\frac{m_3}{m_2} \quad a_{23} = m_2 m_3 \quad c = \frac{m_2^3 m_3^3}{m}.$$

The following proposition summarizes these results:

**Proposition 3.** *Suppose  $r_{ij}(t)$  are Puiseux series with order vector  $(\alpha_{12}, \alpha_{31}, \alpha_{23}) = (1, 1, 0)$  which satisfy equations  $g_1 = g_2 = 0$  identically in  $t$ . Moreover, suppose that  $r_{12}(t) = t$ . Then there are just two possibilities for  $r_{31}(t)$  and  $r_{23}(t)$ . Either*

- I.  $r_{31}(t) = a_{31}t + b_{31}t^2 + \dots$ ,  $r_{23}(t) = a_{23} + b_{23}t^2 + \dots$  with coefficients satisfying (16) and (17), or
- II.  $r_{31}(t) = a_{31}t + \hat{b}_{31}t^4 + \dots$ ,  $r_{23}(t) = a_{23} + b_{23}t^2 + \dots$  with coefficients and parameter,  $c$ , satisfying (19) and (18).

In either case, all of the indicated coefficients are nonzero.

**4.3. Non-existence of Series for  $d$  and  $\rho$ .** In this section, it will be shown that it is impossible to extend the Puiseux series solutions of Proposition 3 by finding series  $d(t), \rho(t)$  satisfying  $g_4 = g_5 = g_{10} = 0$  and  $d(t)$  not identically 0. Cases I and II will be treated separately. Also, it will turn out that case I splits into three subcases, but they can all be treated together.

For case I, make the substitutions  $r_{12}(t) = t, r_{31}(t) = a_{31}t + b_{31}t^2$ ,  $r_{23}(t) = a_{23} + b_{23}t^2$  in equations (11). The result is equations of the form:

$$(20) \quad \begin{aligned} G_4 &= C_{00}(t) + C_{20}(t)d^2 + C_{11}(t)d\rho + C_{40}(t)d^4 + C_{31}(t)d^3\rho = 0 \\ G_5 &= D_{00}(t) + D_{20}(t)d^2 + D_{02}(t)\rho^2 = 0 \\ G_{10} &= E_{00}(t) + E_{20}(t)d^2 + E_{11}(t)d\rho + E_{40}(t)d^4 = 0 \end{aligned}$$

where the coefficients  $C_{ij}, D_{ij}, E_{ij}$  are very complicated polynomials in  $t$  which also involve the parameters  $m_i, a_{ij}, b_{ij}, c, \omega_0, \omega_1$ . Equations (20) can be viewed as three equations for the variables  $d, \rho, t$ . The goal is to show that there do not admit Puiseux series solutions  $d(t) = a_d t^{\alpha_d} + \dots, a_d \neq 0$  and  $\rho(t)$  (the possibility that  $\rho(t)$  is identically 0 is open at this point). The method will be to show that it is impossible to find even the leading exponent  $\alpha_d$  and coefficient  $a_d$  for  $d(t)$ . This is in the spirit of proposition 2 but it turns out that it can be done in a completely elementary way.

Fortunately, it will turn out that only the lowest-order terms in  $t$  of the polynomials  $C_{ij}, D_{ij}, E_{ij}$  will be needed. Let  $C_{ij}(t) = c_{ij}t^{k_{ij}} + \dots$  where  $c_{ij} \neq 0$  and introduce a similar notation for  $D_{ij}, E_{ij}$ . After finding all these lowest-order coefficients and the corresponding exponents for  $t$  one can define a much-simplified system of equations:

$$(21) \quad \begin{aligned} H_4 &= c_{00} + c_{20}t^8d^2 + c_{11}t^8d\rho + c_{40}t^{16}d^4 + c_{31}t^{16}d^3\rho \\ H_5 &= d_{00} + d_{20}t^8d^2 + d_{02}t^4\rho^2 \\ H_{10} &= e_{00}t^q + e_{20}td^2 + e_{11}td\rho + e_{40}t^9d^4 \end{aligned}$$

where the coefficients  $c_{ij}, d_{ij}, e_{ij}$  are rational functions in the parameters  $m_i, a_{ij}, b_{ij}$  which are given in the Appendix. The exponent  $q$  is either 0, 1, 2 depending on the choice of the parameters  $m_i, c$ . For the different choices, the coefficients  $e_{00}$  are

different. However, the argument given below works for all three cases. The only thing one really needs to know about the coefficients is that they are nonzero when the masses are positive and when  $b_{31} \neq 0$  (which is the condition defining case I).

Suppose one has Puiseux series  $d(t), \rho(t)$  solving (20) and suppose  $d(t)$  is not identically zero. Then since Puiseux series form a field and since  $d(t), E_{ii}(t)$  are nonzero elements of this field, one can solve equations  $G_{10} = 0$  for  $\rho(t)$ . Then substituting into the other two equations gives two equations  $\tilde{G}_4(d, t) = \tilde{G}_5(d, t) = 0$  for  $d(t)$ . However, only the lowest-order terms in  $t$  of the coefficients of these new equation are needed and these can be found by carrying out the elimination of  $\rho$  in (21) instead of (20).

Solving  $H_{10} = 0$  gives:

$$\rho = -\frac{e_{00} t^q + e_{20} t d^2 + e_{40} t^9 d^4}{e_{11} t d}$$

Substituting into  $H_4, H_5$  and clearing denominators gives:

$$\begin{aligned} K_4 &= c_{00}e_{11} - c_{11}e_{00} t^{7+q} + (c_{20}e_{11} - c_{11}e_{20}) t^8 d^2 + c_{31}e_{00} t^{15+q} d^2 \\ &\quad + (c_{40}e_{11} - c_{31}e_{20} - c_{11}e_{40}) t^{16} d^4 - c_{31}e_{40} t^{24} d^6 \\ K_5 &= d_{02}e_{00}^2 t^{2+2q} + d_{00}e_{11}^2 d^2 + 2d_{02}e_{00}e_{20} t^{3+q} d^2 + d_{02}e_{20}^2 t^4 d^4 \\ &\quad + d_{20}e_{11}^2 t^8 d^4 + 2d_{02}e_{00}e_{40} t^{11+q} d^4 + 2d_{02}e_{20}e_{40} t^{12} d^6 + d_{02}e_{40}^2 t^{20} d^8. \end{aligned}$$

If there is a Puiseux solution  $d(t) = a_d t^{\alpha_d} + \dots$ ,  $a_d \neq 0$  then the vector  $\alpha = (a_d, 1)$  must be an inward normal to edges of the classical Newton polygons  $Q_4, Q_5$  of  $K_4, K_5$ . Now plotting the exponents  $(k_d, k_t)$  for the monomials of  $K_4$  gives a polygon  $Q_4$  which has only one bottom edge – the one containing  $(0, 0), (2, 8), (4, 16), (6, 24)$ . This is true no matter which value is used for  $q$ . This edge has inward normal  $\alpha = (-4, 1)$  so the only possible leading exponent for  $d(t)$  is  $\alpha_d = -4$ .

Substituting  $d = a_d t^{-4} + \dots$  and taking the lowest-order terms in  $t$  gives reduced equations:

$$\begin{aligned} L_4 &= c_{00}e_{11} + (c_{20}e_{11} - c_{11}e_{20}) a_d^2 + (c_{40}e_{11} - c_{11}e_{40}) a_d^4 - c_{31}e_{40} a_d^6 = 0 \\ L_5 &= d_{02}e_{20}^2 a_d^4 + 2d_{02}e_{20}e_{40} a_d^6 + d_{02}e_{40}^2 a_d^8 = 0. \end{aligned}$$

To show that these have no common root, take the resultant with respect to  $a_d$ :

$$\begin{aligned} Res(L_4, L_5) &= c_{00}^4 d_{02}^6 e_{11}^8 e_{40}^4 (c_{40}e_{20}^2 - c_{20}e_{20}e_{40} + c_{00}e_{40}^2)^4 \\ &= 2^{16} 3^{30} \frac{c^{300} a_{23}^{14} b_{13}^{46} m_1^{62} m^{300}}{m_2^{266} m_3^{108}} \neq 0. \end{aligned}$$

The discussion of case II is similar. The substitutions  $r_{12}(t) = t, r_{31}(t) = a_{31} t + \hat{b}_{31} t^4, r_{23}(t) = a_{23} + b_{23} t^2$  in equations (11) together with (19) give (20) again, but with different coefficient polynomials. Finding the lowest-order terms in  $t$  of these coefficients gives the simplified system of equations:

$$\begin{aligned} \hat{H}_4 &= \hat{c}_{00} + \hat{c}_{20} t^{10} d^2 + \hat{c}_{11} t^{12} d \rho + \hat{c}_{40} t^{20} d^4 + \hat{c}_{31} t^{22} d^3 \rho \\ (22) \quad \hat{H}_5 &= \hat{d}_{00} + \hat{d}_{20} t^{10} d^2 + \hat{d}_{02} t^6 \rho^2 \\ \hat{H}_{10} &= \hat{e}_{00} + \hat{e}_{20} d^2 + \hat{e}_{11} t^2 d \rho + \hat{e}_{40} t^{10} d^4 \end{aligned}$$

where  $\hat{c}_{ij}, \hat{d}_{ij}, \hat{e}_{ij}$  are given in the Appendix. As before, they are rational functions in the parameters  $m_i, a_{ij}, b_{23}, \hat{b}_{31}$  which are nonzero under the prevailing assumptions.

Solving  $\hat{H}_{10} = 0$  gives:

$$\rho = -\frac{e_{00} + e_{20} d^2 + e_{40} t^{10} d^4}{e_{11} t^2 d}$$

Substituting into  $\hat{H}_4, \hat{H}_5$  and clearing denominators gives:

$$\begin{aligned} \hat{K}_4 &= c_{00}e_{11} - c_{11}e_{00} t^{10} + (c_{20}e_{11} - c_{11}e_{20}) t^{10} d^2 + c_{31}e_{00} t^{20} d^2 \\ &\quad + (c_{40}e_{11} - c_{31}e_{20} - c_{11}e_{40}) t^{20} d^4 - c_{31}e_{40} t^{30} d^6 \\ \hat{K}_5 &= d_{02}e_{00}^2 t^2 + d_{00}e_{11}^2 d^2 + 2d_{02}e_{00}e_{20} t^2 d^2 + d_{02}e_{20}^2 t^2 d^4 \\ &\quad + d_{20}e_{11}^2 t^{10} d^4 + 2d_{02}e_{00}e_{40} t^{12} d^4 + 2d_{02}e_{20}e_{40} t^{12} d^6 + d_{02}e_{40}^2 t^{22} d^8. \end{aligned}$$

This time the Newton polygon  $\hat{Q}_4$  has a unique bottom edge containing  $(0, 0)$ ,  $(2, 10)$ ,  $(4, 20)$ ,  $(6, 30)$  showing that the leading exponent for  $d(t)$  is  $\alpha_d = -5$ .

Making the substitution  $d = a_d t^{-5} + \dots$  and taking the lowest-order terms in  $t$  gives:

$$\begin{aligned} \hat{L}_4 &= c_{00}e_{11} + (c_{20}e_{11} - c_{11}e_{20}) a_d^2 + (c_{40}e_{11} - c_{11}e_{40}) a_d^4 - c_{31}e_{40} a_d^6 = 0 \\ \hat{L}_5 &= d_{02}e_{20}^2 a_d^4 + 2d_{02}e_{20}e_{40} a_d^6 + d_{02}e_{40}^2 a_d^8 = 0. \end{aligned}$$

which are exactly the same as for case I. With the new values of the coefficients, the resultant is:

$$\begin{aligned} Res(\hat{L}_4, \hat{L}_5) &= c_{00}^4 d_{02}^6 e_{11}^8 e_{40}^4 (c_{40}e_{20}^2 - c_{20}e_{20}e_{40} + c_{00}e_{40}^2)^4 \\ &= \frac{3^{30} 5^{46}}{2^{18}} m_1^{62} m_2^{326} m_3^{714} (m_2^3 + m_3^3)^{46} \neq 0 \end{aligned}$$

so there are no solutions  $a_d$ .

This completes the proof the theorem 1. Here is a brief summary. If a solution with constant moment of inertia were not a rigid motion then the mutual distances would not be constant. In this case, the algebraic equations satisfied by such solutions which were derived in section 3 would admit solutions with infinitely many different values for the distances  $r_{ij}$  and with the velocity variable  $d \neq 0$ . In that case, there would exist Puiseux series solutions  $r_{ij}(t), d(t), \rho(t)$  all of which are not identically zero, with the possible exception of  $\rho(t)$ . But in sections 4.2 and 4.3, it was found that no such Puiseux series solutions can exist. QED

## 5. APPENDIX

This appendix contains the leading coefficients in  $t$  for the coefficient polynomials  $C_{ij}(t), D_{ij}(t), E_{ij}(t)$  from section 4.3. For case I, the coefficients in 21 are:

$$\begin{aligned}
c_{00} &= (c^{12}m_1^2m^{12})/(m_2^{10}m_3^6) \\
c_{20} &= (2c^{14}b_{31}m_1^2m^{14})/(m_2^{14}m_3^5) \\
c_{11} &= (18c^{11}a_{23}b_{31}^2m_1^2m^{11})/(m_2^{10}m_3^4) \\
c_{40} &= (c^{16}b_{31}^2m_1^2m^{16})/(m_2^{18}m_3^4) \\
c_{31} &= (54c^{13}a_{23}b_{31}^3m_1^2m^{13})/(m_2^{14}m_3^3) \\
d_{00} &= (c^6a_{23}m_1m^6(m_1m_2^3 + m_2^3m_3 + (m_1 + m_2)m_3^3))/(m_2^8m_3^4) \\
d_{20} &= (c^8a_{23}b_{31}m_1m^8(m_1m_2^3 + m_2^3m_3 + (m_1 + m_2)m_3^3))/(m_2^{12}m_3^3) \\
d_{02} &= (-3c^4a_{23}b_{31}m_1m^4)/m_2^5 \\
e_{20} &= (-10c^9b_{31}m_1^2m^9)/(m_2^7m_3^4) \\
e_{11} &= (9c^6a_{23}b_{31}^2m_1^2m^6)/(m_2^3m_3^3) \\
e_{40} &= (-4c^{11}b_{31}^2m_1^2m^{11})/(m_2^{11}m_3^3)
\end{aligned}$$

The form of the term  $e_{00}t^q$  depends on the choice of parameters, as will now be described.

The first three terms of the coefficient  $E_{00}(t)$  lead to a version of formula  $H_{10}$  in (21) with  $e_{00}$  replaced by  $e_{000} + e_{001}t + e_{002}t^2$ . It turns out that any one of these could be the leading term  $e_{00}t^q$ . The first possibility is that  $e_{00}t^q = e_{000}$  where

$$e_{000} = c^3m^4(m_2m_3 - a_{23})(5cmm_2m_3 - a_{23}(m_2^3m_3^3 + 4cm))/(m_2^6m_3).$$

This will be the leading coefficient provided it is nonzero. Recall that the assumption of case I is that  $a_{23} \neq m_2m_3$ . So  $e_{000} = 0$  only if

$$5cmm_2m_3 - a_{23}(m_2^3m_3^3 + 4cm) = 0$$

Taking the resultant of this expression and the formula for  $a_{23}^2$  in (16):

$$mc(mc - m_2^3m_3^3)(16mc - m_2^3m_3^3) = 0.$$

The expression in the first parentheses is nonzero for case I and so

$$(23) \quad c = \frac{m_2^3m_3^3}{16m}$$

is a necessary condition for  $e_{000} = 0$ .

Assuming that (23) holds and  $e_{000} = 0$  one finds

$$\begin{aligned}
e_{001} &= m_2^3m_3^9m(-57m_2^4m_3^4 + 8m_1^3m_2m_3(m_2^3 + m_3^3)) \\
&\quad + a_{23}(147m_2^3m_3^3 - 32m_1^3(m_2^3 + m_3^3))/(65536m_1) \\
e_{002} &= -\frac{9}{16384}m_1m_2^2m_3^9m(m_2^3 + m_3^3)(-58a_{23} + 13m_2m_3)
\end{aligned}$$

It turns out that  $e_{002} \neq 0$ . In fact, the resultant of this expression and the formula for  $a_{23}^2$  in (16) is  $-\frac{165}{4}m_2^3m_3^3$  which is nonzero. Thus, the leading term  $e_{00}t^q$  is either  $e_{001}t^1$  or  $e_{002}t^2$  as claimed.

Case II does not involve analysis of subcases. The coefficients in (22) are always:

$$\begin{aligned}
\hat{c}_{00} &= m_1^2 m_2^{26} m_3^{30} \\
\hat{c}_{20} &= -5m_1^2 m_2^{21} m_3^{35} (m_2^3 + m_3^3) \\
\hat{c}_{11} &= \frac{225}{2} m_1^2 m_2^{10} m_3^{26} (m_2^3 + m_3^3)^2 \\
\hat{c}_{40} &= \frac{25}{4} m_1^2 m_2^{16} m_3^{40} (m_2^3 + m_3^3)^2 \\
\hat{c}_{31} &= \frac{3375}{4} m_1^2 m_2^5 m_3^{31} (m_2^3 + m_3^3)^3 \\
\hat{d}_{00} &= m_1 m_2^{11} m_3^{15} (m_1 m_2^3 + m_2^3 m_3 + m_1 m_3^3 + m_2 m_3^3) \\
\hat{d}_{20} &= -\frac{5}{2} m_1 m_2^6 m_3^{20} (m_2^3 + m_3^3) (m_1 m_2^3 + m_2^3 m_3 + m_1 m_3^3 + m_2 m_3^3) \\
\hat{d}_{02} &= -\frac{15}{2} m_1 m_2 m_3^{11} (m_2^3 + m_3^3) \\
\hat{e}_{00} &= -\frac{3}{4m_2} m_1^3 m_3^7 m (m_2^3 + m_3^3)^2 \\
\hat{e}_{20} &= -50m_1^2 m_2^{13} m_3^{21} (m_2^3 + m_3^3) \\
\hat{e}_{11} &= \frac{225}{4} m_1^2 m_2^2 m_3^{12} (m_2^3 + m_3^3)^2 \\
\hat{e}_{40} &= 50m_1^2 m_2^8 m_3^{26} (m_2^3 + m_3^3)^2
\end{aligned}$$

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