COUNTING RELATIVE EQUILIBRIUM CONFIGURATIONS OF THE FULL TWO-BODY PROBLEM

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Abstract. Consider a system of two rigid, massive bodies interacting according to their mutual gravitational attraction. In a *relative equilibrium* motion, the bodies rotate rigidly and uniformly about a fixed axis in $\mathbb{R}^3$. This is possible only for special positions and orientations of the bodies. After fixing the angular momentum, these relative equilibrium configurations can be characterized as critical points of a smooth function on configuration space. The goal of this paper is to use Morse theory and Lusternik-Schnirelmann category theory to give lower bounds for the number of critical points when the angular momentum is sufficiently large. In addition, the exact number of critical points and their Morse indices are found in the limit as the angular momentum tends to infinity.

1. Introduction

Consider the problem of n rigid, massive bodies in $\mathbb{R}^3$ moving under the influence of their mutual gravitational attraction. Simplifying to the case of point masses provides a good model when the masses are far away from one another or are spherically symmetric. But when asymmetrical masses interact at comparatively close range, dissipative effects can lead to changes in the orbits and the rotational motions. Such forces may lead to a decrease in the total energy of the system, leaving the total angular momentum unchanged (at least according to some simple models [16, page 163]). From this point of view it is interesting to look for local minima or critical points of the energy for a given angular momentum.

In the point mass n-body problem, Smale showed that the critical points of the energy on such a fixed angular momentum level are the relative equilibrium states [21, 22] and the same holds true for the full n-body problem. For such motions, the entire configuration rotates uniformly around some axis in $\mathbb{R}^3$ with the centers of mass moving on circles around the axis and with the bodies rotating simultaneously to maintain phase locking. If such a motion arises due to energy dissipation, it should be a local minimum of the energy and not just a critical point. In [14] it was shown that such

Date: December 12, 2017.

2000 *Mathematics Subject Classification.* 70F10, 70F15, 37N05, 70G40, 70G60.

Key words and phrases. Celestial mechanics, rigid bodies.

Research supported by NSF grant DMS-1712656 .

energy minimizing relative equilibria are impossible for $n \geq 3$. The goal of the present paper is to look in more detail at the case $n = 2$.

When $n = 1$, there is no gravitational interaction, and we are reduced to the well-studied case of a free rigid body. In this case, the relative equilibrium motions are the steady rotations around the principal axes of inertia and the minimum energy motions are rotations around the axis of maximum moment of inertia.

For $n = 2$ the problem is much more complicated and there is a substantial literature about it. Lagrange studied the librations of the moon in its motion around the earth [8]. If we think of the earth as spherical and unaffected by the moon, then we have the problem of a rigid body moving in a central force field and a phase locked moon would be a relative equilibrium even though the earth is not phase locked. Using some approximations to the potential, Lagrange found 24 relative equilibria in the generic case of a body with three distinct principle moments of inertia. These are such that the three principle axes are along the radius, tangent and normal to the orbit. The 4 minimum energy solutions have the principle axis of maximal moment of inertia along the normal and the axis of minimal moment of inertia along the radius. The central force problem is also of interest with regard to the motion of earth satellites [3, 5]. More recent references which include further references to the older papers are [2, 4, 15, 23]. Using various approximations to the potential, they again find 24 relative equilibria in the generic case. The approximate potentials used have more discrete symmetries than is present in the real system. In [23], Morse theory is used to estimate the number of relative equilibria for the full potential.

One of the few papers to treat the case of two arbitrary bodies is Maciejewski's [9]. In the limit of large distances between generic bodies, the principle axes of each body again align with the radius, tangent and normal to the orbit. Maciejewski counts 36 solutions but this does not take account of the choice of orientations along the principle axes. Using the same counting method as above, the number should be $24 \times 24 = 576$. In Section 3 we rederive Maciejewski's result from the point of view of critical points for fixed angular momentum and compute the Morse indices of the 576 solutions. It turns out that there are 16 minima corresponding to cases where both principle axes of maximal moment of inertia point along the normal and those of minimal moment of inertia point along the radius. In addition to studying the limiting case, we use Morse theory and Lusternik-Schnirelman theory to show that for moderately large angular momenta, there must be at least 32 relative equilibria if all are nondegenerate and at least 12 with no nondegeneracy assumptions. The number of minima is at least 2. Better lower bounds are possible for bodies with discrete symmetries.

2. Relative equilibria as critical points

In this section we introduce our notation and state some information about relative equilibria of n-rigid-body problem referring to [14] for the proofs. Consider a collection of n rigid, massive bodies in $\mathbb{R}^3$. Each body has its own body coordinate system where it occupies a compact subset $\mathcal{B}_i \subset \mathbb{R}^3$ which is the support of a mass measure dm_i on $\mathcal{B}_i$, $i = 1, \ldots, n$. Denote the i-th body coordinate system by $Q_i \in \mathbb{R}^3$. The total mass of the i-th body is

$$m_i = \int_{\mathcal{B}_i} dm_i$$

and we assume $m_i > 0$. Assume that the center of mass is at the origin in body coordinates, i.e.,

$$\int_{\mathcal{B}_i} Q_i dm_i = 0.$$

The symmetric 3×3 *inertia matrix* of $\mathcal{B}_i$ is given by

$$I_i = \int_{\mathcal{B}_i} \left(|Q_i|^2 \mathbb{I} - Q_i Q_i^T\right) dm_i \tag{1}$$

where $\mathbb{I}$ is the 3×3 identity matrix. It will be assumed that the matrices I_i are all invertible which excludes point masses and one-dimensional mass distributions.

We describe the position and orientation of the body in the inertial coordinates, $x \in \mathbb{R}^3$ by a time-dependent Euclidean transformation $E_i(t)$ where

$$x(t, Q_i) = E_i(t)(Q_i) = A_i(t)Q_i + q_i(t) \qquad Q_i \in \mathcal{B}_i. \tag{2}$$

The rotation matrix $A_i(t) \in \mathbf{SO}(3)$ gives the orientation of the body while $q_i(t) \in \mathbb{R}^3$ is the inertial center of mass.

The positions and orientations of all n bodies is represented by $Z = (q_1, \ldots, q_n, A_1, \ldots, A_n) \in \mathbb{R}^{3n} \times \mathbf{SO}(3)^n$ and we define the *configuration space* as the open subset of $\mathbb{R}^{3n} \times \mathbf{SO}(3)^n$ where the bodies are disjoint

$$\tilde{\mathcal{U}} = \{Z : E_i(\mathcal{B}_i) \cap E_j(\mathcal{B}_j) = \emptyset, i \neq j\}.$$

To describe the gravitational interaction introduce the *Newtonian potential* function. For each pair of indices (i, j), $i \neq j$, there is a mutual potential

$$U_{ij}(q_i, q_j, A_i, A_j) = \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \frac{dm_i \, dm_j}{|q_i - q_j + A_i Q_i - A_j Q_j|}$$

and then the Newtonian potential is given by

$$U(Z) = \sum_{i<j} U_{ij}.$$

$U(Z)$ is a well-defined, smooth, positive function $U : \tilde{\mathcal{U}} \to \mathbb{R}$. We are calling $U(Z)$ the Newtonian potential, but the potential energy of the system is actually $-U(Q)$.

The velocity of the point (2) is given by

$$\dot{x}(t, Q_i) = \dot{q}_i(t) + \dot{A}_i(t)Q_i = v_i(t) + A_i(t)\hat{\Omega}_i(t)Q_i.$$

Here v_i is the velocity of the center of mass and

$$\hat{\Omega}_i(t) = A_i^{-1}(t)\dot{A}_i(t)$$

is the antisymmetric angular velocity matrix with respect to body coordinates. There is a corresponding *angular velocity vector* $\Omega_i \in \mathbb{R}^3$ such that $\hat{\Omega}_i u = \Omega_i \times u$ for all vectors $u \in \mathbb{R}^3$. The angular velocity and orientation matrices are related by

$$\dot{A}_i(t) = A_i(t)\hat{\Omega}_i(t).$$

In addition to the position variables q_i, A_i, velocity variables v_i, Ω_i will be used on the phase space $T\tilde{\mathcal{U}}$. We will also use the notation $P = (Z, \dot{Z})$ for points of $T\tilde{\mathcal{U}}$.

We will not need the equations of motion here but they can be found in [9, 14]. We note, however, that these equations have the usual symmetries and constants of motion. The translational symmetry $q_i \mapsto q_i + c$, $c \in \mathbb{R}^3$ leads to the constancy of the total momentum vector

$$p_{tot} = m_1 v_1 + \ldots + m_n v_n.$$

Assume without loss of generality that $p_{tot} = 0$. Then the center of mass is constant and we may assume it lies at the origin of the inertial system. This leads to a translation-reduced phase space $T\mathcal{U}$ where

$$\mathcal{U} = \{Z \in \tilde{\mathcal{U}} : m_1 q_1 + \ldots + m_n q_n = 0\}.$$

We have $\dim \mathcal{U} = 6n - 3$ and $\dim T\mathcal{U} = 12n - 6$.

There is also a rotational symmetry. If $R \in \mathbf{SO}(3)$ then the rotated configuration RZ has centers of mass Rq_i and orientation matrices RA_i, $i = 1, \ldots, n$. In other words $\mathbf{SO}(3)$ acts on $\mathbb{R}^{3n} \times \mathbf{SO}(3)^n$ diagonally from the left. The velocities of the centers of mass are also rotated to Rv_i but the body angular velocities Ω_i are unchanged. The rotational symmetry implies the constancy of the total angular momentum vector in the inertial frame

$$\lambda = \sum_i m_i q_i \times v_i + \sum_i A_i I_i \Omega_i. \tag{3}$$

Finally, since no dissipative forces are being included, the total energy

$$H(Z, \dot{Z}) = T(Z, \dot{Z}) - U(Z)$$

is constant, where $T(Z, \dot{Z})$ is the kinetic energy

$$T(Z, \dot{Z}) = \frac{1}{2}\sum_i m_i |v_i|^2 + \frac{1}{2}\sum_i \Omega_i^T I_i \Omega_i.$$

For a relative equilibrium motion, the configuration of n bodies rotates uniformly around a fixed axis through the origin in space. We can write the angular velocity vector of the rotation as ωe where $e \in \mathbb{R}^3$ is a unit vector specifying the direction of the rotation axis and $\omega > 0$ is the angular speed.

The angular momentum vector is related to the angular velocity vector by $\lambda = \omega I(Z)e$ where

$$I(Z) = \sum_i m_i(|q_i|^2\mathbb{I} - q_i q_i^T) + \sum_i A_i I_i A_i^T \tag{4}$$

is the 3×3 *total inertia matrix* of the whole configuration. For a relative equilibrium, it turns out that e must be an eigenvector of $I(Z)$. The eigenvalue is

$$G_e(Z) = e^T I(Z)e = \sum_i m_i q_i^T K_e q_i + \sum_i e^T A_i I_i A_i^T e \tag{5}$$

where K_e is projection onto the plane orthogonal to e. So we have

$$\lambda_{re} = \omega I(Z)e = \omega G_e(Z)e. \tag{6}$$

Similarly, we find that the total energy of a relative equilibrium motion is

$$H_{re} = \frac{1}{2} G_e(Z)\omega^2 - U(Z). \tag{7}$$

In what follows we will be interested in relative equilibria with a given, nonzero angular momentum vector $\lambda \in \mathbb{R}^3$. Then the rotation axis and angular speed are uniquely determined by

$$e = \frac{\lambda}{|\lambda|} \qquad \omega = \frac{|\lambda|}{G_e(Z)}. \tag{8}$$

Of course not every configuration $Z \in \mathcal{U}$ admits a relative equilibrium motion. Indeed, Z must satisfy some complicated algebraic equations such that, for given angular momentum vector λ, we expect only finitely many relative equilibrium configurations up to symmetry. These equations can be derived directly from the equations of motion, but we do not need to do that here. Instead, we will describe a variational approach to the relative equilibrium equations which derives from their relationship to the problem of critical energy.

Fixing an angular momentum vector, λ, defines a subset of the translation-reduced phase space: $\mathcal{M}_\lambda \subset T\mathcal{U}$. It can be shown that for $\lambda \neq 0$, $\mathcal{M}_\lambda$ is a submanifold of codimension three. Then one can look for minima or, more generally, for critical points of the restriction of the energy function H to $\mathcal{M}_\lambda$. Then we have the following result:

Proposition 1. *Let $\lambda \in \mathbb{R}^3$ be any nonzero vector. A state $P = (Z, \dot{Z})$ is a critical point of the restriction of the total energy function to $\mathcal{M}_\lambda$ if and only if it is a relative equilibrium state.*

This can be seen as an application of the general results of Smale [21]. Smale goes on to characterize the corresponding relative configurations Z as critical points of an *amended potential* function. He eliminates the velocities by fixing Z and then minimizing the energy over the corresponding set of velocities $\dot{Z}$ which give angular momentum λ. Substituting the unique minimal velocities into the energy function gives the amended potential:

$$W_\lambda(Z) = \frac{1}{2}\lambda^T I(Z)^{-1}\lambda - U(Z). \tag{9}$$

$W_\lambda(Z)$ is a smooth function on the configuration space $\mathcal{U}$ whose critical points are exactly the configurations which admit relative equilibrium motions with angular momentum λ. Moreover, it follows from the definition that local minimum energy states $P \in \mathcal{M}_\lambda$ correspond to local minima, $Z \in \mathcal{U}$, of $W_\lambda(Z)$.

Recall that for such a relative equilibrium, λ is an eigenvector of the total inertia tensor $I(Z)$. It follows that at these points the amended potential reduces to the simpler form

$$H_\lambda(Z) = \frac{|\lambda|^2}{2G_e(Z)} - U(Z). \tag{10}$$

This function, which will be called the *critical energy function*, was used by Scheeres in his study of minimal energy configurations [17]. While the two functions agree at the critical points, they are not equal in general. In fact we have

Proposition 2. *For $Z \in \mathcal{U}$ and $\lambda \neq 0 \in \mathbb{R}^3$ we have*

$$H_\lambda(Z) \leq W_\lambda(Z) \tag{11}$$

with equality if and only if λ is an eigenvector of $I(Z)$. Both functions provide lower bounds for the energy of any state $P = (Z, \dot{Z}) \in \mathcal{M}_\lambda$.

It turns out that H_λ provides an alternative variational characterization of the relative equilibrium configurations.

Proposition 3. *The amended potential $W_\lambda(Z)$ and the critical energy function $H_\lambda(Z)$ have the same critical points in $\mathcal{U}$, namely the relative equilibrium configurations for angular momentum λ.*

In addition to being simpler, $H_\lambda(Z)$ lends itself better to the application of topological existence techniques in the next section.

Finally, regarding the question of local minima we have the following result:

Proposition 4. *Suppose Z is a relative equilibrium configuration and hence a critical point for both $H_\lambda(Z)$ and $W_\lambda(Z)$. If Z is a local minimum of $H_\lambda(Z)$ then it is also a local minimum for $W_\lambda(Z)$. Conversely, if Z is a local minimum of $W_\lambda(Z)$ and if $I(Z)$ has the property that its maximal eigenvalue is not repeated, then Z is also a local minimum for $H_\lambda(Z)$.*

Note that, due to the main result of [14], such local minima are possible only for $n \leq 2$. The second statement of the proposition was used in the proof of this fact. It is not known whether the extra hypothesis about non-repeated eigenvalues is really needed. The first statement in the proposition is the one which will be useful below. It follows easily from proposition 2. Indeed, if Z is a local minimum of H_λ then for all Z' sufficiently close to Z we have

$$W_\lambda(Z') \geq H_\lambda(Z') \geq H_\lambda(Z) = W_\lambda(Z)$$

so Z is a local minimum for W_λ.

3. Relative equilibria and Minimal energy solutions for $n = 1, 2$

If there is only one rigid body then there is no gravitational interaction and we have the familiar case of a free rigid body. Fixing the center of mass at the origin leaves only the rotational degrees of freedom. If we choose the body coordinates along principle axes, then the angular momentum and energy are

$$\lambda = I_{11}\Omega_1 + I_{22}\Omega_2 + I_{33}\Omega_3 \qquad H = \frac{1}{2}(I_{11}\Omega_1^2 + I_{22}\Omega_2^2 + I_{33}\Omega_3^2).$$

We will use some of the results of the previous section to derive the well known result that the relative equilibria are rotations around the principle axes in the inertial frame and that rotation around a principle axis of maximal moment of inertia gives a minimum energy motion.

The translation-reduced configuration space is $\mathcal{U} = \mathbf{SO}(3)$. If $Z = A \in \mathcal{U}$ is the orientation matrix of the body, then the total inertia tensor is $I(Z) = AI_1A^T$ where $I_1 = \mathrm{diag}(I_{11}, I_{22}, I_{33})$. This is just the inertia tensor of the rigid body in inertial coordinates. The amended potential is

$$W_\lambda(Z) = \frac{1}{2}\lambda^T I(Z)^{-1}\lambda = \frac{1}{2}\lambda^T A I_1^{-1} A^T \lambda$$

while the critical energy function is

$$H_\lambda(Z) = \frac{|\lambda|^2}{2G_e(Z)} \qquad G_e(Z) = e^T I(Z) e = e^T A I_1 A^T e.$$

We can find the relative equilibria as critical points of either function.

Let $A(t) = AR(t)$ where $R(t) \in \mathbf{SO}(3)$ is any curve of rotations with $R(0) = \mathbb{I}$ and $\hat{\rho} = \dot{R}(0) \in \mathbf{so}(3)$. Then Z is a critical point of W_λ if and only if

$$\lambda^T A \hat{\rho} I_1^{-1} A^T \lambda = 0$$

for all $\hat{\rho} \in \mathbf{so}(3)$ or equivalently

$$(A^T\lambda) \cdot (\rho \times I_1^{-1} A^T \lambda) = \rho \cdot (I_1^{-1} A^T \lambda \times A^T \lambda) = 0$$

for all $\rho \in \mathbb{R}^3$. The means $A^T\lambda$ has to be an eigenvector of I_1^{-1}, that is, a principal axis in the body coordinates. Equivalently, λ is a principle axis in the inertial frame. A similar computation to find critical points of H_λ leads to

$$\rho \cdot (I_1 A^T e \times A^T e) = 0$$

for all $\rho \in \mathbb{R}^3$ which means that $e = \lambda/|\lambda|$ is a principle axis in the inertial frame, as before. For a relative equilibrium motion the energy is given by $H_\lambda(Z)$ and it is clear that choosing the principal axis with maximum moment of inertia $G_e(Z)$ gives the minimum energy.

If the principal moment of inertia I_{ii} are distinct then the only principle axes in body coordinates are the coordinate axes. The matrices A such that A^T maps the unit vector $e = \lambda/|\lambda|$ onto a coordinate axis form six circles in $\mathbf{SO}(3)$ according to whether $A^T e = \pm e_i$, $i = 1, 2, 3$. These are the relative equilibrium configurations. In the quotient space $\mathbf{SO}(3)/\mathbb{S}^1 \simeq \mathbb{S}^2$, where $\mathbb{S}^2$ represents the unit rotation axis in body coordinates, we have six relative equilibrium points $\pm e_i$, $i = 1, 2, 3$.

The two-body problem is more interesting. Using the center of mass condition, we can replace the two position variables q_1, q_2 by the relative position $q = q_2 - q_1 = (x, y, z) \in \mathbb{R}^3$. The configuration is given by $Z = (q, A_1, A_2) \in \mathbb{R}^3 \times \mathbf{SO}(3)^2$ and the configuration space is the open set

$$\mathcal{U} = \{(q, A_1, A_2) : E_1(\mathcal{B}_1) \cap E_2(\mathcal{B}_2) = \emptyset\}.$$

Assume, without loss of generality, that $\lambda = (0, 0, |\lambda|)$ and $e = (0, 0, 1)$. Then the moment of inertia around e is

$$G_e(Z) = \mu\, s^2 + e^T A_1 I_1 A_1^T e + e^T A_2 I_2 A_2^T e \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{12}$$

where $s = \sqrt{x^2 + y^2}$. The Newtonian potential is

$$U(Z) = \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \frac{dm_i\, dm_j}{r} \tag{13}$$

where $r(Z, Q_1, Q_2) = |q + A_2 Q_2 - A_1 Q_1|$.

Due to the complexity of the potential, it is difficult to find the relative equilibria for general mass distributions. Several papers have treated special cases of this problem by making simplifying assumptions about one or both of the bodies or by treating limiting cases where the bodies are far apart. See, for example, [9, 23, 17, 18, 19]. We will discuss two of these simplified problems below and also show how to use Morse theory to estimate the number of critical points of H_λ for general mass distributions assuming $|\lambda|$ is sufficiently large.

First consider the case of two spherical bodies with constant densities and radii R_1, R_2. This was discussed in detail by Scheeres [17] but it is worth describing briefly here as motivation for our approach to the general case. According to Newton, the gravitational attraction is the same as for point masses located at the centers of mass, so

$$U(Z) = \frac{m_1 m_2}{|q|}$$

which depends only on $q = (x, y, z)$ and is independent of the orientation matrices of the bodies. The body inertia matrices simplify to $I_j = \alpha_j \mathbb{I}$ where $\alpha_j = \frac{2 m_j R_j^2}{5}$ and (12) simplifies to

$$G_e = \mu\, s^2 + \alpha_1 + \alpha_2.$$

This is also independent of the orientation matrices but differs from the point mass case where $\alpha_1 = \alpha_2 = 0$. A short calculation shows that critical points of H_λ have $z = 0$ and that s satisfies

$$|\lambda|^2 = \frac{(m_1 + m_2)G_e(s)^2}{s^3}.$$

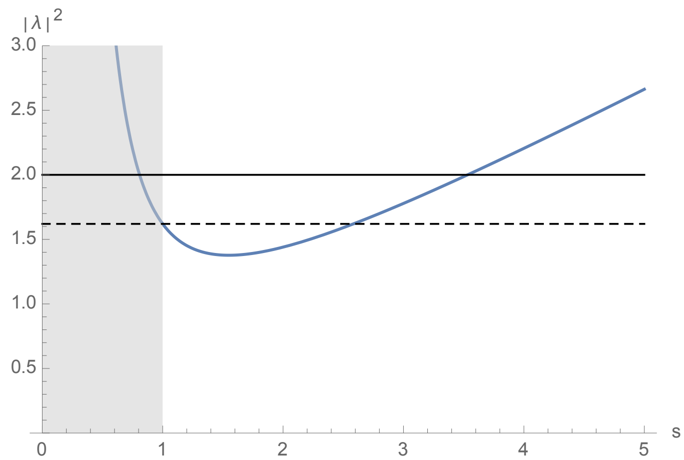

FIGURE 1. Critical points of H_λ for the two spheres problem.

Figure 1 shows a plot of this relation in the $(s, |\lambda|^2)$ plane when $m_1 = m_2 = 1$ and $R_1 = R_2 = \frac{1}{2}$. Fixing $|\lambda|$ determines a horizontal line whose intersection points with the graph represent the corresponding critical points. The shaded region is forbidden by the requirement that the two spheres be disjoint, that is, $s \geq \zeta = R_1 + R_2 = 1$. For $|\lambda|$ small, there are no critical points and the minimum of H_λ is achieved on the boundary with the spheres in contact. For larger $|\lambda|$ two critical points appear and persist until one of them collides with the boundary (dashed line in figure 1). Scheeres calls this the *fission* parameter and shows that an equilibrium with the spheres in contact is no longer possible. For $|\lambda|$ larger than the fission parameter there is a unique critical point in the physical region and it gives the minimum of H_λ. Actually, it is only the values of s and z that are unique. One can still rotate around the z axis and allow arbitrary orientation matrices $A_1, A_2 \in \mathbf{SO}(3)$. Thus, from the point of view of the general problem, each critical point represents a manifold of critical points of $H_\lambda(Z)$ diffeomorphic to $\mathbb{S}^1 \times \mathbf{SO}(3) \times \mathbf{SO}(3)$.

Returning to the problem of two general masses, we begin by construcing a compact subset which contains all of the critical points of H_λ for a given $\lambda = (0, 0, |\lambda|)$ and even includes possible minima with the bodies in contact. Let $\overline{\mathcal{U}}$ denote the closure of $\mathcal{U}$, consisting of configurations where the two bodies are either disjoint or else just barely touch. Note that $\overline{\mathcal{U}}$ is closed but not compact since $|q| = |(x,y,z)| \to \infty$ is possible. Our compact set will be obtained by finding upper bounds for $s = \sqrt{x^2+y^2}$ and $|z|$.

Lemma 1. *For $n = 2$ and $\lambda = (0,0,|\lambda|) \neq 0$, there are positive constants ζ and σ such that all of the relative equilibrium configurations for angular momentum λ are contained in the compact set*

$$K_\lambda = \{Z = (q, A_1, A_2) : q = (x,y,z), |z| \leq \zeta, \sqrt{x^2+y^2} \leq \sigma\,|\lambda|^2\}.$$

Moreover, the infimum of H_λ over $\overline{\mathcal{U}}$ is achieved in K_λ.

Proof. Relative equilibrium configurations are critical points of

$$H_\lambda = \frac{|\lambda|^2}{2G_e(Z)} - U(Z).$$

Let R_1, R_2 be the radii of spheres containing $\mathcal{B}_1, \mathcal{B}_2$ and choose $\zeta = R_1 + R_2$. If $|z| > \zeta$ then one of the bodies is strictly above the other in the sense of z-coordinates. It is easy to see that with this assumption, the Newtonian potential satisfies $U_z < 0$ if $z > 0$ and $U_z > 0$ if $z < 0$ independent of the mass distributions within the bodies. Since the moment of inertia $G_e(Z)$ in (12) is independent of z, it follows that in the set $|z| > \zeta$ there are no critical points of H_λ and that H_λ is increasing with respect to $|z|$ there.

From (12) we have the estimates

$$\mu\, s^2 \leq G_e(Z) \leq \mu\, s^2 + \gamma_1 + \gamma_2 \tag{14}$$

where $s = \sqrt{x^2+y^2}$ and where γ_j is the largest eigenvalue of I_j. The directional derivative in the direction of the unit vector $(x,y,0)/s$ (keeping A_i fixed) is

$$D_s G_e(Z) = 2\mu\, s.$$

To estimate the directional derivative of U, first differentiate (13) to get

$$D_s U(Z) = -\int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{dm_i\, dm_j\, s}{r^3}$$

where $r(Z, Q_1, Q_2) = |q + A_2 Q_2 - A_1 Q_1|$. For $|z| \leq \zeta = R_1 + R_2$ it is easy to see that $r^2 \leq (s+\zeta)^2 + (2\zeta)^2$ for all Q_1, Q_2. This gives

$$-D_s U(Z) \geq \frac{m_1 m_2\, s}{((s+\zeta)^2 + (2\zeta)^2)^{\frac{3}{2}}}.$$

Combining this with the lower bound from (14), it follows that

$$\begin{aligned} D_s H_\lambda(Z) &= -\frac{|\lambda|^2}{2G_e(Z)^2} D_s G_e - D_s U(Z) \\ &\geq -\frac{|\lambda|^2}{\mu\, s^3} + \frac{m_1 m_2\, s}{((s+\zeta)^2 + (2\zeta)^2)^{\frac{3}{2}}}. \end{aligned}$$

Both of the terms tend to zero as $s \to \infty$ but the negative first term is $O(s^{-3})$ whereas the positive second term is $O(s^{-2})$. It follows that we can find σ depending only on m_1, m_2, ζ such that $D_s H_\lambda > 0$ in the region

$|z| \leq \zeta, s > \sigma |\lambda|^2$. Hence there are no critical points in this region and H_λ is increasing with respect to s there. □

Using the compactness of K_λ and reasonable assumptions about the mass distribution, we can show existence of a minimizer for H_λ, possibly with the bodies in contact. For large enough angular momenta, the minimum will occur with no contact and provides a minimal energy relative equilibrium solution.

Theorem 1. *Suppose $n = 2$ and that the mass distributions are such that the Newtonian potential extends continuously to $\partial\mathcal{U}$. Then for every $\lambda \neq 0$ there is a point $Z \in K_\lambda \subset \overline{\mathcal{U}}$ where $H_\lambda(Z)$ achieves its infimum over $\overline{\mathcal{U}}$. If $|\lambda|$ is sufficiently large, then this minimum is achieved at a relative equilibrium configuration $Z \in \mathcal{U}$. In this case Z also gives the infimum of $W_\lambda(Z)$ over $\overline{\mathcal{U}}$ and so the corresponding relative equilibrium state is an energy minimizer in $\mathcal{M}_\lambda$.*

Proof. With these assumptions, H_λ is continuous on the compact set K_λ of lemma 1 and must achieve a minimum there. By the lemma, this is actually the infimum over $\overline{\mathcal{U}}$.

If $|\lambda|$ is large, we need to see that the minimum of H_λ on $\overline{\mathcal{U}}$ does not occur on the boundary. But $\partial\mathcal{U}$ is compact, so there are constants $g > 0$ and $u > 0$ such that

$$G_e(Z) \leq g \qquad U(Z) \leq u \qquad H_\lambda(Z) \geq \frac{|\lambda|^2}{2g} - u \qquad\qquad Z \in \partial\mathcal{U}.$$

So we will have $H_\lambda(Z) > 0$ on $\partial\mathcal{U}$ if $|\lambda|^2 \geq 2gu$. It suffices to show that there is some $Z \in \mathcal{U}$ where $H_\lambda(Z) < 0$.

If we take $|z| \leq \zeta$, use the lower bound from (14) and estimate the denominator in (13) as above we find

$$H_\lambda(Z) \leq \frac{|\lambda|^2}{2\mu\, s^2} - \frac{m_1 m_2}{((s+\zeta)^2 + (2\zeta)^2)^{\frac{1}{2}}}.$$

which is negative for all sufficiently large s.

Since the minimal point Z is not on the boundary, it must be a critical point of H_λ, i.e., a relative equilibrium configuration for angular momentum λ. By proposition 3, this critical point is a local minimum of both H_λ and W_λ, so the corresponding relative equilibrium state is an energy minimizer in $\mathcal{M}_\lambda$. □

For general bodies the boundary $\partial\mathcal{U}$ will be very complicated and a study of possible equilibria with the bodies in contact seems intractable. For the two-spheres problem, Scheeres found that such configurations are impossible for sufficiently large angular momenta. It seems that such a result is not possible without restricting the shape of the bodies. For example, two bodies shaped like the letter S would likely have stable equilibrium configurations with the bodies hooked together for arbitrarily large angular momenta $|\lambda|$.

From now on we will ignore the boundary and focus on critical points of H_λ in $\mathcal{U}$. Lemma 1 shows that these critical points must lie in $K_\lambda \cap \mathcal{U}$. However, we also want to avoid considering possible critical points with the bodies very close to one another where the details of the geometry of the bodies might lead to unexpected results. For example, imagine nested spheres or interlocking dumbbells. To avoid this sort of problem we will impose a lower bound $s \geq \sigma_0$ which forces the bodies to be separated in the directions orthogonal to λ.

Lemma 2. *Let C_λ be the compact set*

$$C_\lambda = \{Z = (q, A_1, A_2) : q = (x, y, z), |z| \leq \zeta, \sigma_0 \leq s \leq \sigma\,|\lambda|^2\}$$

where $s = \sqrt{x^2 + y^2}$, $\zeta = R_1 + R_2$ and σ are the constants from lemma 1 and σ_0 is any constant with $\sigma_0 > \zeta$. For $|\lambda|$ sufficiently large, C_λ is a positively invariant set for the negative gradient flow of H_λ. Moreover, by taking $|\lambda|$ larger we can guarantee that the critical points of H_λ in C_λ actually satisfy a stronger lower bound $s \geq c\,|\lambda|^2$ where c is any constant with $0 < c < \frac{m_1+m_2}{(m_1 m_2)^2}$.

Proof. Let K_λ be the compact set of lemma 1. Clearly C_λ is the subset defined by imposing the additional constraint $s \geq \sigma_0$. The proof of lemma 1 shows that the vectorfield $-\nabla H_\lambda(Z)$ is pointing in along the boundaries $|z| = \zeta$ and $s = \sigma|\lambda|^2$ of K_λ. If $|\lambda|$ is sufficiently large, we will show that it also points in on the new boundary component where $s = \sigma_0$.

To see this, note that for fixed $s > \zeta$, the denominator $r(Z, Q_1, Q_2)$ appearing in the integrals for the potential and its derivative satisfies $r \geq |s - \zeta|$ for all Q_1, Q_2. Combining this with the upper bound from (14) gives

$$\begin{aligned} D_s H_\lambda(Z) &= -\frac{|\lambda|^2}{2G_e(Z)^2} D_s G_e - D_s U(Z) \\ &\leq -\frac{|\lambda|^2 \mu\, s}{(\mu\, s^2 + \gamma_1 + \gamma_2)^2} + \frac{m_1 m_2\, s}{(s - \zeta)^3}. \end{aligned}$$

The first term is negative and dominates the second when

$$|\lambda|^2 > \frac{(m_1 + m_2)(\mu\, s^2 + \gamma_1 + \gamma_2)^2}{(s - \zeta)^3}. \tag{15}$$

Setting $s = \sigma_0$ for some constant $\sigma_0 > \zeta$ we see that the inequality will hold for $|\lambda|$ sufficiently large. To prove the last statement we want the inequality to hold for all $s \in [\sigma_0, c|\lambda|^2]$. Let $f(s)$ denote the right-hand side of (15). It is easy to check that $f''(s) > 0$ so the maximum of $f(s)$ on $[\sigma_0, c|\lambda|^2]$ is attained at an endpoint. We have already chosen $|\lambda|$ such that the inequality if valid for $s = \sigma_0$. Setting $s = c|\lambda|^2$ and taking the limit as $|\lambda| \to \infty$, the inequality (15) simplifies to $c < \frac{m_1+m_2}{(m_1 m_2)^2}$. Fixing any such c, the inequality will hold if $|\lambda|$ is sufficiently large. □

The critical energy function is invariant under rotation around the z-axis so it determines a smooth function on the quotient space $\tilde{\mathcal{X}}_\lambda = C_\lambda/\mathbb{S}^1$. We

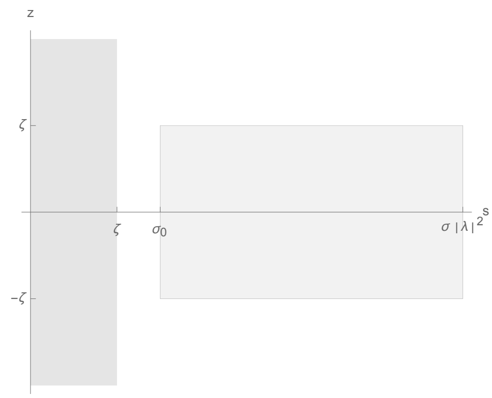

FIGURE 2. $R_\lambda = \{\sigma_0 \le s \le \sigma|\lambda|^2, |z| \le \zeta\}$. If $|\lambda|$ is sufficiently large, the product space $\tilde{\mathcal{X}}_\lambda = R_\lambda \times \mathbf{SO}(3)^2$ is positively invariant for the negative gradient flow of H_λ. We are avoiding the region $s \le \zeta$ where the bodies may come into contact.

can eliminate this symmetry by restricting to $Z = (q, A_1, A_2)$ with $q = (x, 0, z)$, $x = s > 0$. Thus $\tilde{\mathcal{X}}_\lambda$ is diffeomorphic to $R_\lambda \times \mathbf{SO}(3)^2$ where R_λ is the two-dimensional rectangle $\{(s, z) : \sigma_0 \le x \le \sigma|\lambda|^2, |z| \le \zeta\}$ (see figure 2). It is invariant under the negative gradient flow of H_λ if $|\lambda|$ is sufficiently large (where (15) is sufficient). This gives another proof that H_λ must have at least a minimum away from $\partial\mathcal{U}$ but we can use Morse theory to give a better estimate for the number of critical points. Note, however, that by restricting to $\tilde{\mathcal{X}}_\lambda$ we may be missing some critical points with $s < \sigma_0$. For example, comparison of figures 1 and 2 shows that in the two spheres problem there is family of critical points converging to the contact set $s = \zeta$ for a small interval of the parameter $|\lambda|$.

Before introducing the Morse estimates, it is advantageous to take note of a further, discrete symmetry of the problem. Consider a relative equilibrium state $(Z, \dot{Z})$ with angular momentum $\lambda = (0, 0, |\lambda|)$. Let R denote the rotation by by π around the x-axis:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then $(RZ, R\dot{Z})$ is a relative equilibrium motion with angular momentum $R\lambda = -\lambda$. If we combine this with a reversal of all of the velocities, we find a relative equilibrium state $(RZ, -R\dot{Z})$ with angular momentum λ again but with configuration RZ. Using coordinates (x, z, A_1, A_2) on the quotient space $\tilde{\mathcal{X}}$, we have a further action of the group $\mathbb{Z}_2$:

$$R(x, z, A_1, A_2) = (x, -z, RA_1, RA_2).$$

Let $\mathcal{X} = \tilde{\mathcal{X}}/\mathbb{Z}_2$ be the quotient space under this action. Then H_λ determines a smooth function on $\mathcal{X}$. The critical points of this function correspond two-to-one to critical points in $\tilde{\mathcal{X}}$ and to circles of critical points in $\mathcal{U}$. To apply Morse theory we need to find the Betti numbers β_k of $\mathcal{X}$, that is, the ranks of the homology groups with coefficients in some field. Then the *Poincaré polynomial* is $P(t) = \sum_k \beta_k t^k$. In case of degenerate critical points we will use Lusternik-Schnirelman theory which requires knowledge of the cup product structure in the cohomology rings. Specifically, we need to know the cup length, that is, the maximal number of nontrivial factors in a nonzero cup product.

Lemma 3. *Using $\mathbb{Z}_2$ coefficients for homology and cohomology groups, the Poincaré polynomial of $\mathcal{X}$ is $P(t) = (1+t+t^2+t^3)^2 = (1+t)^2(1+t^2)^2$ and its cup length is 5.*

Proof. $\tilde{\mathcal{X}}$ is a trivial disk bundle over $\mathbf{SO}(3)^2$ and it follows that $\mathcal{X}$ is a disk bundle over $\mathbf{SO}(3)^2/\mathbb{Z}_2$. Such a bundle is homotopy equivalent to its base space. In this case we have a homotopy equivalence $\mathcal{X} \simeq \mathbf{SO}(3)^2/\mathbb{Z}_2$. Changing coordinates on $\mathbf{SO}(3)^2$ from (A_1, A_2) to (A_1, B) where $B = A_1^{-1}A_2$, the action becomes trivial on the second factor and we are reduced to $\mathcal{X} \simeq L \times \mathbf{SO}(3)$ where $L = \mathbf{SO}(3)/Z_2$. Now $\mathbf{SO}(3) \simeq \mathbf{RP}(3) \simeq \mathbb{S}^3/\mathbb{Z}_2$ and it follows that the factor L is a lens space $L \simeq \mathbb{S}^3/\mathbb{Z}_4$. Indeed if we view S^3 as the unit quaternions then $\mathbf{SO}(3)$ is the quotient by the subgroup $\{1, -1\}$. The matrix R is the rotation matrix corresponding to the quaternions $\pm i$ and so L can be viewed as the quotient space of the unit quaternions by $\{1, -1, i, -i\}$.

The homology groups and cohomology rings of these spaces are known [6]. For the projective space $\mathbf{RP}(3)$ we have homology groups with $\mathbb{Z}_2$ coefficients $H_k(\mathbf{RP}(3), \mathbb{Z}_2) \simeq \mathbb{Z}_2$, $0 \le k \le 3$. The corresponding Betti numbers are all 1 and the Poincaré polynomial is $1 + t + t^2 + t^3$.

The cohomology rings $H_k(\mathbf{RP}(3), \mathbb{Z}_2)$ are also isomorphic to $\mathbb{Z}_2$ and the cup product structure makes $H^*(\mathbf{RP}(3), \mathbb{Z}_2)$ into a truncated polynomial ring $\mathbb{Z}_2[\alpha]/\alpha^4$ where α is the generator of the first cohomology. The cup length is 3 since $\alpha \cup \alpha \cup \alpha \neq 0$ is the generator of H^3.

The lens space $L \simeq \mathbb{S}^3/\mathbb{Z}_4$ has $\mathbb{Z}_4$ homology and cohomology isomorphic to $\mathbb{Z}_4$, $0 \le k \le 3$ [6, page 251]. A calculation using the universal coefficient theorem shows that the $\mathbb{Z}_2$ homology and cohomology are isomorphic to $\mathbb{Z}_2$. So the Poincaré polynomial with $\mathbb{Z}_2$ coefficients is again $1 + t + t^2 + t^3$. However, the cup product structure is different. Using $\mathbb{Z}_4$ coefficients the generators of $H^k(L, \mathbb{Z}_4)$, $k = 0, \ldots, 3$ are $1, \alpha, \beta, \alpha \cup \beta$ respectively and $\alpha \cup \alpha = 2\beta \in H^2(L, \mathbb{Z}_4)$. Using $\mathbb{Z}_2$ coefficients, we have the same generators, but now $\beta \cup \beta = 0$. So the cup length is only 2.

For product spaces, it follows from the Künneth formulas that the Poincaré polynomials multiply and the cup lengths add (at least when using coefficients in a field), so the lemma follows. □

Combining this with theorem 1 gives an estimate for the minimal number of relative equilibria for sufficiently large λ.

Theorem 2. *Suppose $n = 2$ and let $\lambda \neq 0$ be large enough that the vectorfield induced by $-\nabla H_\lambda$ on $\mathcal{X}$ points in on the boundary. If the critical point in $\mathcal{X}$ are all nondegenerate then there are at least 32 relative equilibria up to rotation around λ. Without assuming nondegeneracy, there must be at least 12 relative equilibria up to rotation around λ.*

Proof. The Poincaré polymomial of $\mathcal{X}$ is $P(t) = (1 + t + t^2 + t^3)^2$. With the assumption of nondegenerate critical points, Morse theory shows that the number of critical point of H_λ in $\mathcal{X}$ is at least the sum of the Betti numbers, that is, $P(1) = 16$ [12]. Since $\mathcal{X}$ is a quotient under the action of $\mathbb{Z}_2$ and of rotations around λ, there must be at least 32 circles of relative equilibria in $\mathcal{U}$. Without assuming nondegeneracy we have to use Lusternik-Schnirelmann category as a lower bound for the number of critical points in $\mathcal{X}$ [7]. The category of $\mathcal{X}$ is at least one more than the cup length, that is, at least 6, so we have at least 12 circles of relative equilibria in $\mathcal{U}$. □

The actual number of relative equilibria can be much larger. For example, when the two bodies are far apart, the mutual potential is well approximated by the first few terms of the Legendre expansion. In [9], Maciejewski uses a continuation argument to construct relative equilibria where the two bodies are far apart. He uses a different reduction, does not fix the angular momentum and does not consider the connection with Morse theory. So we will give a brief derivation of the results for the limiting system here. In the end we will see that there are 576 relative equilibria.

When the separation of the bodies, as measured by the vector $q = q_2 - q_1$, is much larger than the sizes of the bodies we can do a series expansion of the potential:

$$
\begin{aligned}
U(q, A_1, A_2) &= \frac{m_1 m_2}{r} + \frac{1}{2r^3}(m_2 \operatorname{trace} I_1 + m_1 \operatorname{trace} I_2) \\
&\quad - \frac{3}{2r^5}(m_2 q^T A_1 I_1 A_1^T q + m_1 q^T A_2 I_2 A_2^T q) + O(r^{-4})
\end{aligned} \tag{16}
$$

where $r = |q|$. Using the rotational symmetry as above, we may assume $q = (s, 0, z)$ and $r^2 = s^2 + z^2$. Assume also that $e = e_3 = (0, 0, 1)$ and $\lambda = |\lambda| e$ with $|\lambda|$ sufficiently large. We will look for the critical points of

$$
H_\lambda(q, A_1, A_2) = \frac{|\lambda|^2}{2G_e(q, A_1, A_2)} - U(q, A_1, A_2)
$$

on $R_\lambda \times \mathbf{SO}(3)^2$, where

$$
G_e(q, A_1, A_2) = \mu\, s^2 + e_3^T A_1 I_1 A_1^T e_3 + e_3^T A_2 I_2 A_2^T e_3.
$$

To study the limiting problem as $|\lambda| \to \infty$. Let $\epsilon = |\lambda|^{-2}$ and introduce rescaled variables

$$
\hat{s} = \epsilon s \quad \hat{z} = \epsilon z \quad \hat{q} = \epsilon q \quad \hat{r} = \epsilon r
$$

and a rescaled critical energy function

$$F(\hat{s},\hat{z},A_1,A_2,\epsilon) = \epsilon^{-1}H_\lambda(\hat{s}/\epsilon,\hat{z}/\epsilon,A_1,A_2).$$

Using (16) we find

$$F(\hat{s},\hat{z},A_1,A_2,\epsilon) = F_0(\hat{s},\hat{z},\epsilon) + \epsilon^2 F_2(\hat{s},\hat{z},A_1,A_2) + O(\epsilon^4) \tag{17}$$

where

$$\begin{aligned} F_0 =& \frac{1}{2\mu\,\hat{s}^2} - \frac{m_1 m_2}{\hat{r}} - \frac{\epsilon^2}{2\hat{r}^3}(m_2 \operatorname{trace} I_1 + m_1 \operatorname{trace} I_2) \\ F_2 =& -\frac{1}{\mu^2\,\hat{s}^4}(e_3^T A_1 I_1 A_1^T e_3 + e_3^T A_2 I_2 A_2^T e_3) \\ &+ \frac{3}{2\hat{r}^5}(m_2 \hat{q}^T A_1 I_1 A_1^T \hat{q} + m_1 \hat{q}^T A_2 I_2 A_2^T \hat{q}) \end{aligned}$$

Taking the partial derivatives of F with respect to $\hat{s},\hat{z}$ and then taking the limit as $\epsilon \to 0$ gives:

$$-\frac{1}{\mu\,\hat{s}^3} + \frac{m_1 m_2 \hat{s}}{\hat{r}^3} = 0 \qquad \frac{m_1 m_2 \hat{z}}{\hat{r}^3} = 0$$

which gives a unique solution for the relative position vector:

$$\hat{s} = \hat{r} = \frac{m_1 + m_2}{(m_1 m_2)^2} \qquad \hat{z} = 0 \qquad \hat{q} = \hat{s} e_1. \tag{18}$$

For the limiting problem as $\epsilon \to 0$, the orientation matrices A_1, A_2 will be critical points of the function $F_2(\hat{s}, 0, A_1, A_2)$ with $\hat{s}$ given by (18). The dependence of this function on these matrices is decoupled. Each matrix A_i is critical point of a function $f : \mathbf{SO}(3) \to \mathbb{R}$ of the form:

$$f(A) = c_1 e_1^T A I_i A^T e_1 - c_3 e_3^T A I_i A^T e_3 \tag{19}$$

where $c_1 > 0, c_2 > 0$ are given by

$$c_1 = \frac{3m_j}{2\hat{s}^3} \qquad c_3 = \frac{1}{\mu^2\,\hat{s}^4} \qquad \{i,j\} = \{1,2\}.$$

We may assume without loss of generality that the inertia matrices I_i are diagonal: $I_i = \operatorname{diag}(\alpha_i, \beta_i, \gamma_i)$. For simplicity, we will consider the generic case where the principle moments of inertia $\alpha_i, \beta_i, \gamma_i$ are distinct.

Lemma 4. *Let $f : \mathbf{SO}(3) \to \mathbb{R}$ be any function of the form (19) where $A \in \mathbf{SO}(3)$, $c_i > 0$, and e_i are the standard basis vectors in $\mathbb{R}^3$. Furthermore, assume that $I_i = \operatorname{diag}(\alpha_i, \beta_i, \gamma_i)$ with $0 < \alpha_i < \beta_i < \gamma_i$. Then f has exactly 24 critical points, all nondegenerate. The critical matrices A are those which leave the coordinate axes invariant.*

The Morse indices of the critical points can be described as follows. Suppose $A^T e_1 = \pm e_j$, $A^T e_2 = \pm e_k$ and $A^T e_3 = \pm e_l$ where $\{j,k,l\} = \{1,2,3\}$. If $j < k < l$ the index is 0, if $k < j < l$ or $j < l < k$ the index is 1, if $k < l < j$ or $l < j < k$ the index is 2, and if $l < k < j$ the index is 3. There are four critical points of each of these six types.

Proof. Let $u_1 = A^T e_1$ and $u_3 = A^T e_3$. Then $u_1, u_3 \in \mathbb{R}^3$ are orthogonal unit vectors and they uniquely determine $A \in \mathbf{SO}(3)$. We have

$$f(u_1, u_3) = c_1 u_1^T I_i u_1 - c_3 u_3^T I_i u_3.$$

Viewing this as a function on $\mathbb{R}^3 \times \mathbb{R}^3$, we will find critical points subject to the constraints

$$g_1 = u_1 \cdot u_1 = 1 \qquad g_2 = u_1 \cdot u_3 = 0 \qquad g_3 = u_3 \cdot u_3 = 1.$$

Set

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \lambda_3 \nabla g_3$$

where λ_i are Lagrange multipliers. The critical points satisfy

$$c_1 I_i u_1 = \lambda_1 u_1 + \lambda_2 u_3 \qquad - c_3 I_i u_3 = \lambda_3 u_3 + \lambda_2 u_1.$$

Since u_i are orthogonal, we have $\lambda_2 = 0$ and these equations reduce to the condition that both u_1 and u_3 be eigenvectors of the diagonal matrix I_i. In other words, $u_1 = \pm e_j$ and $u_3 = \pm e_l$ for some $j, l \in \{1, 2, 3\}$. There are four choices of sign and six choices for j, l. Then it follows that $A^T e_2 = \pm e_k$ with $\{j, k, l\} = \{1, 2, 3\}$ with this last sign uniquely determined.

The Hessian quadratic form of the constrained system can be found by restricting the 6×6 matrix

$$D\nabla f - \lambda_1 D\nabla g_1 - \lambda_2 D\nabla g_2 - \lambda_3 D\nabla g_3$$

to the tangent space to the constraint manifold. We have

$$\lambda_1 = c_1 e_j^T I_i e_j \qquad \lambda_2 = 0 \qquad \lambda_3 = c_3 e_l^T I_i e_l.$$

Note that $e_j^T I_i e_j = \alpha_i, \beta_i$ or γ_i according to whether $j = 1, 2$ or 3 and similarly for $e_l^T I_i e_l$. Now a simple computation shows that the 6×6 Hessian matrix is block diagonal with 3×3 blocks

$$2c_1(I_i - e_j^T I_i e_j \mathbb{I}) \qquad - 2c_3(I_i - e_l^T I_i e_l \mathbb{I}).$$

The tangent space to the constraint manifold at the critical point is spanned by the following vectors in $\mathbb{R}^6$:

$$v_1 = (e_k, 0) \qquad v_2 = (0, e_k) \qquad v_3 = (e_l, \pm e_j)$$

where the sign in the last vector depends on the sign choices for u_1, u_3. The matrix of the restricted Hessian in this basis is

$$2 \begin{bmatrix} c_1(e_k^T I_i e_k - e_j^T I_j e_j) & 0 & 0 \\ 0 & c_3(e_l^T I_i e_l - e_k^T I_i e_k) & 0 \\ 0 & 0 & (c_1 + c_3)(e_l^T I_i e_l - e_j^T I_i e_j) \end{bmatrix}.$$

The lemma follows from the observation that the signs of the three diagonal entries are the same as those of $k - j, l - k, l - j$ respectively. □

Given a function with nondegenerate critical points on a manifold, the *Morse polynomial* is

$$M(t) = \sum_k \nu(k) t^k$$

where $\nu(k)$ is the number of critical points of Morse index k. Then the results of Lemma 4 can be summarized by saying that the Morse polynomial of $f : \mathbf{SO}(3) \to \mathbb{R}$ is

$$M(t) = 4 + 8t + 8t^2 + 4t^3 = 4(1+t)(1+t+t^2).$$

Finally, we can describe the critical point structure of the critical energy function for the limiting problem as $|\lambda| \to \infty$ and, by perturbation, for $|\lambda|$ sufficiently large.

Theorem 3. *Choose $\sigma_0 > \zeta$. If $|\lambda|$ sufficiently large, then up to rotation around λ, the critical energy function has exactly 576 critical points with $s \geq \sigma_0$, all nondegenerate. The corresponding Morse polynomial is*

$$M(t) = 16 + 64t + 128t^2 + 160t^3 + 128t^4 + 64t^5 + 16t^6 = 16(1+t)^2(1+t+t^2)^2.$$

Proof. We look for critical points of the rescaled critical energy function $F(\hat{s}, \hat{z}, A_1, A_2)$ of (17) for $\epsilon > 0$ sufficiently small. The a priori estimates $c|\lambda|^2 \leq s \leq \sigma|\lambda|^2$, $|z| \leq \zeta$ from lemma 2 give estimates $c \leq \hat{s} \leq \sigma$, $|\hat{z}| \leq \zeta\epsilon$. The restriction of $F(\hat{s}, \hat{z}, A_1, A_2, \epsilon)$ to this set is a smooth function of all variables. It follows that as $\epsilon \to 0$, the critical points converge to the critical points of the limiting problem at $\epsilon = 0$.

For the limiting problem, $\hat{s}, \hat{z}$ are uniquely determined by (18) and each of the matrices A_i is one of the 24 critical point of a function as in Lemma 4. Altogether we have $24^2 = 576$ critical points for the limiting problem. We will use the implicit function theorem to see that each of these can be continued uniquely to $\epsilon > 0$ sufficiently small and that the Morse indices of the corresponding critical points are given by the sum of the Morse indices for the matrices A_1, A_2 as in Lemma 4.

We will use constrained variables u_1^i, u_3^i instead of the matrices A_i as in the proof of the lemma. Then we have a function $F(\hat{s}, \hat{z}, u_1^1, u_3^1, u_1^2, u_3^2, \epsilon)$ with constraints. The gradient has the form

$$\nabla F = (\nabla_1 F_0 + O(\epsilon^2), \epsilon^2 \nabla_2 F_2 + O(\epsilon^4))$$

where ∇_1 is the two-dimensional partial gradient with respect to $(\hat{s}, \hat{z})$ and ∇_2 is the twelve-dimensional gradient with respect to the variables u_j^i.

The Hessian has a corresponding block structure

$$D\nabla F = \begin{bmatrix} D_1 \nabla_1 F_0 + O(\epsilon^2) & O(\epsilon^2) \\ O(\epsilon^2) & \epsilon^2 D_2 \nabla_2 F_2 + O(\epsilon^4) \end{bmatrix}.$$

The matrix $D_1 \nabla_1 F_0$ is diagonal and positive definite and the computations in the proof of lemma 4 show that after applying the constraints, the matrix $D_2 \nabla_2 F_2$ restricts to a nondegenerate 6×6 matrix. Therefore we can apply the implicit function theorem to get unique continuation.

It remains to check the Morse indices of the corresponding critical points. If P is a block diagonal matrix with a 2×2 block $\mathbb{I}$ and a 12×12 block $\epsilon^{-1}\mathbb{I}$ then

$$P^T D\nabla F P = \begin{bmatrix} D_1 \nabla_1 F_0 + O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & D_2 \nabla_2 F_2 + O(\epsilon^2) \end{bmatrix}.$$

Now we can take the limit as $\epsilon \to 0$ to see that the Morse index will be the sum of the contributions from $(\hat{s}, \hat{z})$, A_1 and A_2. Since $D_1\nabla_1 F_0$ is positive definite, we just get the sum of the indices from Lemma 4. Equivalently, the Morse polynomials for A_1, A_2 can just be multiplied to get $M(t)$ as claimed. $\square$

It is interesting to compare the Morse theoretical estimates to the actual number of critical points in the limiting case, or equivalently, to compare the Morse polynomial $M(t)$ of Theorem 3 to the Poincaré polynomial $P(t)$ of Lemma 3. The Morse inequalities imply that $M(t) = P(t)+(1+t)R(t)$ where $R(t)$ is a polynomial with nonnegative coefficients. Taking into account the extra $\mathbb{Z}_2$ action used in Lemma 3 we need to divide the $M(t)$ in Theorem 3 by 2. So we have

$$M(t) = 8(1+t)^2(1+t+t^2)^2 \qquad P(t) = (1+t)^2(1+t^2)^2.$$

The Morse inequalities hold with

$$R(t) = 7 + 23t + 38t^2 + 38t^3 + 23t^5 + 7t^5.$$

There is a large gap between the exact count of 576 critical points for the limiting case and the Morse estimate of 32. It would be interesting to explore numerically the bifurcations of the relative equilibria of some irregularly shaped bodies as the angular momentum is decreased from near infinity to more moderate values.

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