Instructions

- This is a 50 minutes exam. No calculators, computers cellphones allowed.

- You are allowed to use a formula sheet: one side of an A4 size paper. Your formula sheet cannot be shared with colleagues and must be submitted with your exam!

- This exams contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. Please turn off cell phones.

- Do not give numerical approximations to quantities such as $\sin(5)$, $\pi$, or $\sqrt{2}$. However, you should simplify $\cos(\frac{\pi}{2}) = 0$, $e^0 = 1$, and so on.

- Put your answers in the boxes!! (except for item 2-a)

The following rules apply:

- Show your work, in a reasonably neat and coherent way, in the space provided. All answers must be justified by valid mathematical reasoning. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.

- Mysterious or unsupported answers will not receive full credit. Your work should be mathematically correct and carefully and legibly written.

- A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

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1. (10 points) Given a plane $P_1$ with the equation $x + 2y - 4z = 4$, and another plane $P_2$ with the equation $x + y - z = 1$, find their line of intersection (denoted by $\mathcal{L}$).

**Solution:** Just solve the linear system:

$$
\begin{align*}
x + 2y - 4z &= 4 \\
x + y - z &= 1
\end{align*}
$$

Your answer should be

$$
\{(x, y, z) \in \mathbb{R}^3 | (x, y, z) = (-2 - 2t, 3 + 3t, t), t \in \mathbb{R}\},
$$

i.e., the line passing through $(-2, 3, 0)$ in the direction $<-2, 3, 1>$. 

$\mathcal{L} =$
2. (15 points) Given the function
\[ f(x, y, z) = (z^2 + 1) \ln[x + \sin(y)], \]
find:

a) (5 points) The gradient of \( f \) (i.e., find \( \nabla f(x, y, z) \)).

Solution:
\[
\nabla f(x, y, z) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle (z^2 + 1)(x + \sin(y)), \frac{(z^2 + 1) \cos(y)}{(x + \sin(y))}, 2z \ln(x + \sin(y)) \rangle
\]

b) (10 points) The directional derivative of \( f \) at the point \( (1, 0, 0) \) in the direction
\[ u = \frac{1}{3} < 1, 2, -2 >. \]

Solution:

As the vector \( \vec{u} \) is unitary, we get
\[
D_u f(1, 0, 0) = \frac{1}{3} \langle 1, 1, 0 \rangle \cdot \langle 1, 2, -2 \rangle = 1
\]
3. (15 points) Given the $C^2(\mathbb{R}^2)$ function

$$f(x, y) = \frac{e^x}{1+y} + \frac{e^{x(10^3+y^2+y^4)}}{(10^3+y^2+y^4)},$$

Compute:

a) (10 points) $\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} f \right)$.

**Solution:** We did something similar in a quiz. As the function is $C^2$ we can use Clairaut’s theorem to reduce the problem to

$$\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f \right)$$

Notice that

$$\left( \frac{\partial}{\partial x} f \right) = \frac{e^x}{1+y} + e^{x(10^3+y^2+y^4)}$$

And we conclude that

$$\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f \right) = -\frac{e^x}{(1+y)^2} + x(2y+4y^3)e^{x(10^3+y^2+y^4)}$$

\[\begin{array}{c}
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} f \right) = \\
\end{array}\]

a) (5 points) $\frac{\partial^2}{\partial x^2} f$.

**Solution:** Using \[\] we get

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f \right) = \frac{\partial}{\partial x} \left( \frac{e^x}{1+y} + e^{x(10^3+y^2+y^4)} \right)$$

$$= \frac{e^x}{1+y} + (10^3+y^2+y^4)e^{x(10^3+y^2+y^4)}.$$

\[\begin{array}{c}
\frac{\partial^2}{\partial x^2} f = \\
\end{array}\]
4. (15 points) The parametric equation of two lines \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are given below.

\[
\begin{align*}
\mathcal{L}_1 : & \quad x - 2 = \frac{y + 1}{2} = 3 - z, \\
\mathcal{L}_2 : & \quad (x, y, z) = (-s, -1, 2 - s), \quad s \in \mathbb{R}.
\end{align*}
\]

a) (5 points) Show that these two lines do not intersect.

**Solution:** If they do then the second component of both should be -1, hence \( t = 0 \), forcing the point to be \((2, -1, 3)\) (using \( \mathcal{L}_1 \) equation). But then, using first component of \( \mathcal{L}_2 \), we might have \( s = -2 \), while last component says \( s = -1 \) (absurd).

b) (10 points) We actually know a bit more about these two lines: we know that they are skew lines\(^1\) (no intersection point and not parallel). Find their distance.

**Solution:**

First you find a plane parallel to \( \mathcal{L}_1 \) containing \( \mathcal{L}_2 \). You get

\[-x + y + z = 1.\]

Now you pick a point in both lines and find the vector connecting them:

\[P_1 \in \mathcal{L}_1, \quad P_1 = (2, -1, 3), \quad P_2 \in \mathcal{L}_2, \quad P_2 = (0, -1, 2).\]

Hence, \( \overrightarrow{P_2 P_1} = < 2, 0, 1 > \).

Now the distance will be the length of the projection onto the normal vector.

\[
\text{distance} = \frac{|< 2, 0, 1 > \cdot < -1, 1, 1 >|}{\sqrt{3}} = \frac{\sqrt{3}}{3}.
\]

\( \text{Distance}(\mathcal{L}_1, \mathcal{L}_2) = \)

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\(^1\)You don’t need to prove it, no worries
5. (20 points) Given the surface \( S \)
\[
x^2 + y^2 + z^2 + 2xz = 1
\]
a) (5 points) Find \( \frac{\partial z}{\partial x} \)

**Solution:** Define \( F(x, y, z) = x^2 + y^2 + z^2 + 2xz - 1 \); the surface \( S \) corresponds to the level set \( \{ F = 0 \} \). Now we know that

\[
\frac{\partial z}{\partial x} = -\frac{\left( \frac{\partial F}{\partial x} \right)}{\left( \frac{\partial F}{\partial z} \right)} = -2(x + z) / 2(x + z) = -1
\]

b) (5 points) Find \( \frac{\partial z}{\partial y} \)

**Solution:**

\[
\frac{\partial z}{\partial y} = -\frac{\left( \frac{\partial F}{\partial y} \right)}{\left( \frac{\partial F}{\partial z} \right)} = -\frac{2y}{2(x + z)} = -\frac{y}{x + z}
\]

c) (10 points) Find the tangent plane to \( S \) at \( (x_0, y_0, z_0) = \left( \frac{1}{2}, 0, \frac{1}{2} \right) \).

**Solution:** Using items a and b we have that

\[
\frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = 0,
\]

Hence, the equation for the tangent plane is

\[
z - \frac{1}{2} = -1(x - \frac{1}{2}) + 0(y) \quad \Rightarrow \quad z + x = 1.
\]
6. (15 points) Let
\[ f(x, y) = [(x + y) + 2\sin(y)]^4, \]
where
\[ x = ts, \quad y = t^2 + \frac{s\pi}{4}. \]
Calculate \( \frac{\partial f}{\partial t} \) at \( t = 0, \ s = 2 \).

Solution: You need to use the Chain Rule:
\[ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}. \tag{2} \]

At \( t = 0, \ s = 2 \) we have \( x(t = 0, s = 2) = 0 \) and \( y(t = 0, s = 2) = \frac{\pi}{2} \). Hence
\[ \frac{\partial f}{\partial x}(0, 2) = 4[(x + y) + 2\sin(y)]^3 \bigg|_{(0, \frac{\pi}{2})} = 4\left(\frac{\pi}{2} + 2\right)^3. \]

Likewise,
\[ \frac{\partial f}{\partial y}(0, 2) = 8\cos(y)[(x + y) + 2\sin(y)]^3 \bigg|_{(0, \frac{\pi}{2})} = 0. \]

At \( (t, s) = (0, 2) \) we get
\[ \frac{\partial f}{\partial t}\bigg|_{(t, s) = (0, 2)} = s \bigg|_{(t, s) = (0, 2)} = 2, \quad \frac{\partial f}{\partial t}\bigg|_{(t, s) = (0, 2)} = 2t \bigg|_{(t, s) = (0, 2)} = 0 \]
Therefore, plugging in (2) we get
\[ \frac{\partial f}{\partial t} = 8\left(\frac{\pi}{2} + 2\right)^3. \]

7. (10 points) Given the function
\[ f(x, y) = \frac{x^2 + xy^3 + y^6}{x^2 + y^6}, \quad (x, y) \neq (0, 0). \]

a) (5 points) Does the limit \( \lim_{(x, y) \to (0, 0)} f(x, y) \) exist?

Solution: It does not exist. The proof of this fact is standard: pick two different paths converging to zero along with the evaluated function \( f \) has different limits.

path #1 : \( x = s, y = 0 \).

path #2 : \( x = t^3, y = t \).

We get
path #1

\[ \lim_{s \to 0} f(s, 0) = \lim_{s \to 0} \frac{s^2}{s^2} = 1. \]

path #2

\[ \lim_{t \to 0} f(t^3, t) = \lim_{t \to 0} \frac{3t^6}{2t^6} = \frac{3}{2}. \]

As these numbers are different the limit is not a well defined quantity and we are done.

b) (5 points) Given function

\[ F(x, y) = \begin{cases} 
  f(x, y), & \text{whenever } (x, y) \neq (0, 0); \\
  0, & \text{whenever } (x, y) = (0, 0). 
\end{cases} \]

Find its domain. Furthermore, find the points in \( \mathbb{R}^2 \) where \( F(x, y) \) is continuous.

**Solution:** The domain is \( (x, y) \in \mathbb{R}^2 \), since the function \( f \) can take values for any of these points.

We know that the function \( F \) is not continuous at \( (0,0) \), since the equality \( \lim_{(x,y) \to (0,0)} F(x, y) = F(0, 0) \) does not hold. We need to analyse the case \( (x, y) \neq (0, 0) \). First notice that both

\[ p(x, y) := x^2 + xy^3 + y^6 \]

and \( q(x, y) := x^2 + y^6 \) are continuous functions, therefore their quotient \( \frac{p(x, y)}{q(x, y)} \) will be continuous as long as the \( q(x, y) \neq 0 \). However, \( q(x, y) = x^2 + y^6 = 0 \) if, and only if, \( x = y = 0 \). As \( (x, y) \neq (0, 0) \) it will never happen. Therefore \( F \) is continuous for all \( (x, y) \neq (0, 0) \).

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\[ ^2 \text{This is a funny statement commonly used in math lingo. Logically it means "it is necessary and sufficient that".} \]