Instructions

- This is a 15 minutes quiz. No calculators, computers cellphones allowed. YOU CANNOT CONSULT BOOKS, NOR NOTES, NOR FRIENDS, NOR COLLEAGUES...

1. (6 points) Given two vectors $\mathbf{a} = <1, 3, 4>$ and $\mathbf{b} = <-1, 3, -2>$ find:

i. Find the equation of the line parallel to $\mathbf{a}$ and passing through the point $(2, -2, 0)$.

Solution: The parametrized equation is

$$(2, -2, 0) + t(1, 3, 4).$$

ii. Find the equation of the plane parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$ and passing through the point $(2, -2, 0)$.

Solution: As I said in class, to find a plane you need to find a vector (normal to this plane) and a point on this plane. For the normal $\mathbf{N}$ you need the cross product:

$$\mathbf{N} = \mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix} = <-18, -2, 6>.$$ 

And we conclude that the equation of the plane is

$$-18(x - 2) - 2(y + 2) + 6z = 0 \implies -18x - 2y + 6z = -32.$$ 

2. (3 points) Find the distance of the point $(1, -2, -1)$ to the plane $x - 4z = 2$.

Solution: You can of course go for the approach "use the formula..." but I explain below a more geometric way (also discussed in class).

i. First you find a point on the plane, which is not hard: set two of the variables equal to zero and solve for the remaining one: for instance, $A(2, 0, 0)$ is in the plane.

ii. now you find the vector going from $A(2, 0, 0)$ to $(1, -2, -1)$. This gonna be the vector

$$\mathbf{v} = <-1, -2, -1>.$$ 

iii. Now you compute the normal to the plane (we discussed it a few days ago):

$$\mathbf{N} = <1, 0, -4>.$$
iv. Now you do as in the first quiz: you project v onto N and find the length of this projected vector \( P_N(v) \). It will give you the distance. We have
\[
P_N(v) = \frac{N \cdot v}{|N|^2} N \implies |P_N(v)| = \frac{|N \cdot v|}{|N|}.
\]
Now we compute \( N \cdot v \):
\[N \cdot v = 3,
\]
and \( |N| \):
\[|N| = \sqrt{1 + 0^2 + (-4)^2} = \sqrt{17}.
\]
Hence,
\[\text{distance} = \frac{3}{\sqrt{17}},\]
and we are done.

3. (3 points) As we discussed in class last week, the lines
\[
L_1 : \quad x = 1 - t, \quad y = -2 + 3t \quad z = 4 - t
\]
\[
L_2 : \quad x = 2s \quad y = 3 + s \quad z = -3 + 4s
\]
are skew lines, i.e., they have no point of intersection and are not parallel lines. Find their distance.

Solution: Well... this exercise is very similar to an exercise discussed in class except for a “+” sign. I will skip the part that shows that these lines are skew lines and focus on finding their distance.

i. First you find a point on each plane, which is not hard: set \( t = 0 \) to get \( A_1(1, -2, 4) \in L_1 \) and \( s = 0 \) to get \( A_2(0, 3, -3) \in L_1 \).

ii. Now you find the vector going from \( A_1A_2 = <1, -5, 7> \).

iii. As you (should) know, \( L_1 \) is of the form \( A_1 + t < -1, 3, -1 > \) for \( t \in \mathbb{R} \), and \( L_2 \) is of the form \( A_2 + s < 2, 1, 4 > \) for \( s \in \mathbb{R} \). If we consider a plane that contains lines in both the directions \( < -1, 3, -1 > \) and \( < 2, 1, 4 > \) we get a plane with normal \( N = < -1, 3, -1 > \times < 2, 1, 4 > = < 13, 2, -7 > \).

iv. Now you do as in the first quiz: you project \( A_1A_2 \) onto \( N \) and find the length of this projected vector \( P_N(A_1A_2) \). It will give you the distance. We have
\[
P_N(v) = \frac{N \cdot A_1A_2}{|N|^2} N \implies |P_N(A_1A_2)| = \frac{|N \cdot A_1A_2|}{|N|}.
\]
Now we compute \( N \cdot A_1A_2 \):
\[N \cdot A_1A_2 = 13 \cdot 1 + 2 \cdot (-5) - 7 \cdot 7 = -46,
\]
and \( |N| := \sqrt{222} \). Hence,
\[\text{distance} = \frac{|-46|}{\sqrt{222}} = \frac{46}{\sqrt{222}},\]
and we are done.

\(^1\)Example 10, Section 12.5, page 830
\(^2\)Check this!!