Instructions

- This is a 15 minutes quiz. No calculators, computers cellphones allowed. YOU CANNOT CONSULT BOOKS, NOR NOTES, NOR FRIENDS, NOR COLLEAGUES...

1. (6 points) Find the absolute maximum and minimum of \( f(x, y) = x^2 + y^2 - 2x \) on \( D \), where \( D \) a closed triangular region with vertices \((2,0), (0,2)\) and \((0,-2)\).

Solution 1
First you find critical points: 
\[
\nabla f(x, y) = \langle 2x - 2, 2y \rangle = \langle 0, 0 \rangle
\]
\( \implies \) \((x, y) = (1, 0)\). Notice that the critical point is INSIDE the triangular domain, so we must take it into account.

Now we analyze the function on the boundary. By the way, how do we write the boundary of this triangle? Notice that we only need the segments of line connecting

1. \((2,0)\) and \((0,2)\)
2. \((2,0)\) and \((0,-2)\)
3. \((0,2)\) and \((0,-2)\)

It is not hard to write these equations. I will write this in the first you find the vector connecting two points \(A(x_0, y_0)\) and \(B(x_1, y_1)\). The vector \(AB = \langle x_1-x_0, y_1-y_0 \rangle\), hence the vector \(v(t) = (x_0, y_0) + t < x_1-x_0, y_1-y_0 >\), for \(t \in [0,1]\), does connect \((x_0, y_0)\) (that’s the value of \(v(0)\)) to \((x_1, y_1)\) (that’s the value of \(v(1)\)). Alright, so let’s to back and write the segments that make up the boundary of the triangle:

1. \((2,0)\) and \((0,2)\) \(\implies v_1(t) = (2, 0) + t(-2, 2) = (2 - 2t, 2t)\);
2. \((2,0)\) and \((0,-2)\) \(\implies v_2(t) = (2, 0) + t(-2, -2) = (2 - 2t, -2t)\);
3. \((0,2)\) and \((0,-2)\) \(\implies v_3(t) = (0, 2) + t(0, -4) = (0, 2 - 4t)\),

for \(t \in [0,1]\).

Now we need to investigate the maximum and minimum of the function \(f\) over these segments, i.e., \(f[v_j(t)]\) for \(j = 1, 2, 3\) and \(t \in [0,1]\).

1. \(f[v_1(t)] = f(2 - 2t, 2t) = (1 - 2t)^2 + 4t^2 + 1 \implies f'(t) = -4(1 - 2t) + 8t = 0 \implies t = 1/4;\)
2. \(f[v_2(t)] = f[(2 - 2t, -2t) = (1 - 2t)^2 + 4t^2 + 1 \implies f'(t) = -4(1 - 2t) + 8t = 0 \implies t = 1/4;\)
3. \(f[v_3(t)] = f(0, 2 - 4t) = 4(1 - 2t)^2 \implies f'(t) = -16(1 - 2t) = 0 \implies t = 1/2;\)

Hence we just need to check the value of the function at these critical points AND the value on the boundary of each segment. We end up with this table:
The value in red (resp., in blue) is the maximum value (resp., minimum value) taken by $f$ on $D$.

2. (4 points) Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ where 

\[
z = \ln(3x + 2y), \quad x = s \sin(t), \quad y = t \cos(s).
\]

Solution 2  

We need to apply the chain rule. In general I would skip details, but I have the impression that most of you are not very confortable using the chain rule (even in 1 variable!!!). We know that

\[
\frac{\partial}{\partial t} z = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.
\]

Likewise,

\[
\frac{\partial}{\partial s} z = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}.
\]

Let’s compute each term:

1. $\frac{\partial x}{\partial t} = \frac{d}{dt}(s \sin(t)) = s \cos(t)$
2. $\frac{\partial y}{\partial t} = \frac{d}{dt}(t \cos(s)) = \cos(s)$
3. $\frac{\partial x}{\partial s} = \sin(t)$
4. $\frac{\partial y}{\partial s} = -t \sin(s)$

now we plug this all back into equations (1) and (2):

\[
\frac{\partial}{\partial t} z = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{3 \cos(t) + 2 \cos(s)}{3x + 2y}.
\]

\[
\frac{\partial}{\partial s} z = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{3 \sin(t) - 2t \sin(s)}{3x + 2y}.
\]

Are we done?...No!!! We want a solution in terms of $s$ and $t$! But we still have $x$ and $y$ in the solutions! Let’s get rid of them. Recall that $x = ssin(t)$ and $y = tcos(s)$.

We conclude that

\[
\frac{\partial}{\partial t} z = \frac{3s \cos(t) + 2 \cos(s)}{3s \sin(t) + 2t \cos(s)}.
\]

\[
\frac{\partial}{\partial s} z = \frac{3 \sin(t) - 2t \sin(s)}{3s \sin(t) + 2t \cos(s)}.
\]