Instructions

• This is a 15 minutes quiz. No calculators, computers cellphones allowed. YOU CANNOT CONSULT BOOKS, NOR NOTES, NOR FRIENDS, NOR COLLEAGUES...

1. (5 points) Find the area of the bounded region \( D \) between the curves \( y = x - x^3 \) and \( y = -3x \) intersected with the half space \( x \geq -1 \).

Solution:

![Graph](image)

Figure 1: In blue, the curve \( f(x) = -3x \); in red, the curve \( g(x) = x - x^3 \); in black, the curve \( x = -1 \). We are interested in the regions A and B.

In the graph above we want It is not hard to parametrize this domain, but we have to split it in two pieces: \( D_1 \) and \( D_2 \), where

\[
A = \{(x,y)\mid -1 \leq x \leq 0, x - x^3 \leq y \leq -3x\}.
\]

\[
B = \{(x,y)\mid 0 \leq x \leq 2, -3x \leq y \leq x - x^3\}.
\]

Now the area becomes

\[
A = \int_D dA = \int_A dA + \int_B dA = \int_{-1}^{0} \int_{x-x^3}^{-3x} dydx + \int_{0}^{2} \int_{-3x}^{x-x^3} dydx = \int_{-1}^{0} [-4x + x^3]dx + \int_{0}^{2} [4x - x^3]dx = \frac{23}{4}.
\]
2. (5 points) (2 of them are bonus points) Find the volume above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 1 \).

Solution:

The plots of each of these surfaces are given below:

(a) \( z = \sqrt{x^2 + y^2} \)

(b) \( z = \sqrt{1 - x^2 - y^2} \)

First you need to figure out a domain of parametrization. Therefore we need to find an upper surface and a lower surface. As you can see in the picture above \( S_1 : z = \sqrt{x^2 + y^2} \) is the lower surface while \( S_2 : z = \sqrt{1 - x^2 - y^2} \) is the upper surface. First we find their intersection

\[
S_1 \cap S_2 = \{ \sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2} \} = \{ x^2 + y^2 = 1/2 \}.
\]

We can use the "shadow" under these surfaces to parametrize this object, which corresponds to the circle

\[
D = \{ (x, y) | x^2 + y^2 \leq 1/2 \} \quad \text{Polar coord.} \quad \{ (r, \theta) | r \leq \frac{\sqrt{2}}{2}, \quad 0 \leq \theta \leq 2\pi \}.
\]

we integrate as

\[
V = \int \int_D \int \frac{1 - x^2 - y^2}{\sqrt{x^2 + y^2}} \, dV = \int \int_D \left[ \sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2} \right] \, dA
\]

and now, in polar coordinates, we have,

\[
V = \int_0^{2\pi} \int_0^{\sqrt{2}/2} \left[ \sqrt{1 - r^2} - r \right] r \, dr \, d\theta = 2\pi \left( -\frac{(1 - r^2)^{3/2}}{3} - \frac{r^3}{3} \right) \bigg|_0^{\sqrt{2}/2}
\]

\[
= \pi \left( \frac{2 - \sqrt{2}}{3} \right).
\]