Some Properties Of Finite Morphisms On Double Points

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Abstract. For a finite morphism $f : X \to Y$ of smooth varieties such that f maps X birationally onto $X' = f(X)$, the local equations of f are obtained at the double points which are not triple. If $\mathcal{C}$ is the conductor of X over X', and $D = \mathrm{Sing}(X') \subset X'$, $\Delta \subset X$ are the subschemes defined by $\mathcal{C}$, then D and Δ are shown to be complete intersections at these points, provided that $\mathcal{C}$ has "the expected" codimension. This leads one to determine the depth of local rings of X' at these double points. On the other hand, when $\mathcal{C}$ is reduced in X, it is proved that X' is weakly normal at these points, and some global results are given. For the case of affine spaces, the local equations of X' at these points are computed.

Keywords: generic projections, double points, singular locus, conductor, complete intersection, weak normality, seminormality, local equations

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0. Introduction

While generic projections enjoy significant properties, part of these properties may be derived from rather mild assumptions on arbitrary finite morphisms. This was the philosophy of the second author to initiate this work. Let X be a projective smooth variety of dimensions r and let $\pi : X \to \mathbb{P}^m$ be a strongly generic projection with $r + 1 \leq m \leq 2r$. Outside a closed subset of X of low dimension, the local canonical form of π is known (see [18], sec. 12). These canonical forms enable one to study the local properties of π, the local structure of $\pi(X)$ and that of the singular locus of $\pi(X)$. As an example, most of the approaches to deal with the Andreotti-Bombieri-Holm conjecture, which claims that $\pi(X)$ is a weakly normal variety, have been of local nature, i.e., based on the local canonical forms of π (see [6], Theo. 3.7, [1], Theo. 2.7, [25], Theo. 3.2). Thus it is tempting to look for rather general situations when one still may get canonical forms of morphisms.

Another related feature, is the structure of the singular loci of maps. An old result due to Enriques ([5], page 8), states that if a surface has a pinch point, its singular locus is smooth at this point. This has been generalized as follows: if $\pi : X \to \mathbb{P}^m$ is a strongly generic projection, and if $y = \pi(x)$ is a pinch point, then $\mathrm{Sing}(\pi(X))$ is smooth at y (see [25], Prop. 2.14). Again one can study this question for rather general morphisms. On the other hand, even for strongly generic projections, $\mathrm{Sing}(\pi(X))$ at triple points is not Cohen-Macaulay unless $m = r + 1$, i.e., $\pi(X)$ is a hypersurface (see [21], Prop. 4.6, [26], Cor. 2.7). Since the hypersurface case is extensively studied(see [19], [9]), we will not deal with this case, and will only study the double points which are not triple.

We will consider a finite morphism $f : X \to Y$ of smooth varieties of dimensions r and m respectively, where $r + 1 \leq m \leq 2r$. It is known that every irreducible component of $\mathrm{Sing}(f(X))$ has dimension at least $2r - m$ (e.g., see [2], Theo. 5.1). As for generic projections, one of the basic ingredients to derive certain results is that the singular locus has the least dimension, a similar hypothesis in the case of arbitrary finite morphism leads to some nice results on $\mathrm{Sing}(f(X))$ at the double points. Primarily one needs a suitable scheme structure on $\mathrm{Sing}(f(X))$. A useful scheme structure in this situation is the structure given by the conductor of X in $f(X)$. This provides a scheme structure not only for $D = \mathrm{Sing}(f(X))$ but also for $\Delta = f^{-1}(D)$ (see [9]). Throughout this paper we will use the scheme structure on D and Δ given by the conductor. One of our results is that when D or Δ has the expected dimension, they are both complete intersections at the double points of f. These results meet Kleiman's criterion of not imposing any hypothesis other than the appropriateness of the dimension of the singular locus.

The conductor happens to have an important role in dealing with weak normality too. By a result due to C. Traverso ([22], Lemma 1.3), if $X' = f(X)$ is weakly normal, then the conductor is a reduced subscheme of X. The converse is not true in general. But it turns out that for the double points which are not triple, this condition is sufficient to guarantee weak normality of X' (see Theo. 4.8). The weak normality property needs to be checked at "depth one primes" ([6], Cor. 2.7). In view of this, we have shown that actually, the reducedness of the conductor at only one double point on each irreducible component of the singular locus will still imply the weak normality of X'(see Theo. 4.9). Then using a result of Vitulli, we have given a necessary and sufficient version of Hartshorne's depth-connectivity result for the local rings on X'(see Prop. 4.10). The local defining ideals of double and triple singularities of strongly generic projections and finite presentations related to them are studied in ([21]). We have obtained similar results for the double points provided that the canonical forms of f are given by polynomials rather than formal power series (see Theo. 5.2).

1. Preliminaries

Let B be an integral over-ring of A, the seminormalization of A in B is defined to be

$$\underset{B}{\overset{+}{}}A = \{b \in B \mid \forall x \in \mathrm{Spec}(A), b_x \in A_x + R(B_x)\}$$

where $R(B_x)$ is the Jacobson radical of B_x, while the weak normalization of A in B is defined to be

$$\underset{B}{\overset{*}{}}A = \{b \in B \mid \forall x \in \mathrm{Spec}(A), \exists n \in \mathbb{N} : (b_x)^{p^n} \in A_x + R(B_x)\}$$

If $A = \underset{B}{\overset{+}{}}A$, we say A is seminormal in B and if $A = \underset{B}{\overset{*}{}}A$, A is called weakly normal in B. If B is the integral closure of A in its total ring of quotients, then we say A is SN (resp. WN) if it is SN (resp. WN) in B.

Since $\underset{B}{\overset{+}{}}A$ is contained in $\underset{B}{\overset{*}{}}A$ (and they coincide if char $k = 0$), thus if A is WN in B, it is also SN in B.

A more geometric interpretation of seminormality and weak normality is the following:

The ring $\underset{B}{\overset{+}{}}A$ is always SN in B and is the largest among the subrings A' of B containing A such that:

$i)\forall x \in \mathrm{Spec}(A)$, there exists exactly one $x' \in \mathrm{Spec}(A')$ lying over x,

$ii)$ the canonical homomorphism $k(x) \to k(x')$ is an isomorphism.

In the same way $\underset{B}{\overset{*}{}}A$ is always WN in B and is the largest among the subrings A' of B containing A such that;

$i)\forall x \in \mathrm{Spec}(A)$, there exists exactly one $x' \in \mathrm{Spec}(A')$ lying over x,

$ii)$ the canonical homomorphism $k(x) \to k(x')$ makes $k(x')$ a purely inseparable extension of $k(x)$ (see [11], [6]).

The following two results will be used in section 4 to check seminormality and weak normality of certain schemes.

PROPOSITION 1.1. *For an integral extension $A \subset B$ the following statements are equivalent*

1) A is SN in B.

2) For each b in B, the conductor of A in $A[b]$ is a radical ideal of $A[b]$.

3) A contains each element b of B such that $b^n, b^{n+1} \in A$ for some positive integer n.

4) For a fixed pair of relatively prime integers $e > f > 1$, A contains each element b of B such that $b^e, b^f \in A$.

Proof. (see [10], Prop. 1.4). □

PROPOSITION 1.2. *Let $A \subset B$ be as above, then the following are equivalent*

1) A is WN in B.

2) A is SN in B and every element b in B which satisfies $b^p \in A$ and $pb \in A$ for some prime integer p, belongs to A.

Proof. (see [24], Theo. 1). □

A scheme is called SN (resp. WN), if the stalks are all SN (resp. WN) rings. By a complete intersection, we mean a local ring whose completion is a quotient of a complete regular local ring by a regular sequence. A scheme is locally complete intersection, if the stalks are all complete intersections. (see [12] sec. 21, [8], Ch. II, Remark 8.22.2).

We now will recall some results in algebraic geometry which will be used freely in this paper.

Let k be an algebraically closed field and let Z be a variety over k, $\mathcal{O}_{Z,z}$ the local ring at $z \in Z$. It is known that $\widehat{\mathcal{O}}_{Z,z}$ is reduced, hence (0) is the intersection of prime ideals $P_1, \cdots, P_d$ in $\widehat{\mathcal{O}}_{Z,z}$. These minimal prime ideals are defined to be the branches of Z at z. If P is a branch of Z at z, we say that P is simple if $\widehat{\mathcal{O}}_{Z,z}/P$ is a regular ring, otherwise it is singular. Let X be the normalization of Z, $\nu : X \to Z$ the canonical morphism. The branches of Z at z are in 1-1 correspondence with points $x \in X$ such that $\nu(x) = z$. Namely, x corresponds to the kernel of the homomorphism $\widehat{\mathcal{O}}_{Z,z} \to \widehat{\mathcal{O}}_{X,x}$ (see [15], Theo. A, or [14], (37.6)). Let X and Y be smooth varieties over k of dimensions r and m respectively, where $r + 1 \leq m \leq 2r$. Let $f : X \to Y$ be a finite morphism which is birational onto $X' = f(X)$. Let $y \in X'$, $f^{-1}(y) = \{x_1, \cdots, x_d\}$ and let $\mathcal{O}_{X,f^{-1}(y)}$ be the semilocal ring along the fibre, which is the ring of germs of functions which are regular at x_i; $i =$

$1, \cdots, d$. Consider the local homomorphism $\mathcal{O}_{Y,y} \xrightarrow{f^*} \mathcal{O}_{X,f^{-1}(y)}$ where $\mathcal{O}_{Y,y}$ is the local ring at y. If $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$, by finiteness of f, $\mathcal{O}_{X,f^{-1}(y)}/f^*(\mathfrak{m}_y)\mathcal{O}_{X,f^{-1}(y)}$ is a semilocal Artinian ring, therefore it has finite length as an $\mathcal{O}_{Y,y}$-module. This length is called "the multiplicity of f at y". Since X is the normalization of X', we may call this length "the multiplicity of X' at y". Since any semilocal Artinian ring is a product of Artinian local rings,

$$\mathcal{O}_{X,f^{-1}(y)}/f^*(\mathfrak{m}_y)\mathcal{O}_{X,f^{-1}(y)} \cong \prod_{i=1}^{d}(\mathcal{O}_{X,x_i}/f_i^*(\mathfrak{m}_y)\mathcal{O}_{X,x_i})$$

where $f_i^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x_i}$ is the natural homomorphism. Thus the length of the first module is the sum of the lengths of the factors on the right hand side above. Since any Artinian local ring is complete, we get the isomorphism

$$\mathcal{O}_{X,f^{-1}(y)}/f^*(\mathfrak{m}_y)\mathcal{O}_{X,f^{-1}(y)} \cong \prod_{i=1}^{d}(\widehat{\mathcal{O}}_{X,x_i}/f_i^*(\mathfrak{m}_y)\widehat{\mathcal{O}}_{X,x_i}). \qquad (\star)$$

The length of each factor in the above may be interpreted as "the multiplicity of a branch of X' ". Therefore we can say that the multiplicity of X' at y is the sum of the multiplicities of the branches of X' at y.

From now on, we will only consider the points on X' with multiplicity 2. Since the f_i^* 's are local homomorphisms, the length of each factor of the product in $(\star)$ is at least one. Thus, either $d = 2$ and each of the two factors in $(\star)$ is isomorphic to k, or, $d = 1$ and $\text{length}(\widehat{\mathcal{O}}_{X,x}/f^*(\mathfrak{m}_y)\widehat{\mathcal{O}}_{X,x}) = 2$. In the first case, $\widehat{\mathcal{O}}_{X',y} \cong \widehat{\mathcal{O}}_{Y,y}/\ker(\widehat{f_1^*}) \cap \ker(\widehat{f_2^*})$, where, $\widehat{\mathcal{O}}_{Y,y}/\ker(\widehat{f_i^*}) \cong \widehat{\mathcal{O}}_{X,x_i}$ for $i = 1, 2$, thus X' has two simple branches at y. In the latter case, X' has an analytically irreducible double point at y.

The morphism f determines an exact sequence of sheaves on X'

$$0 \to \mathcal{O}_{X'} \to f_*\mathcal{O}_X \to f_*\mathcal{O}_X/\mathcal{O}_{X'} \to 0.$$

The conductor of X over X' is the annihilator of $f_*\mathcal{O}_X/\mathcal{O}_{X'}$ as an $\mathcal{O}_{X'}$-module. This is a sheaf of ideals in $\mathcal{O}_{X'}$ and naturally lifts to a sheaf of ideals in $\mathcal{O}_X$. We will use $\mathcal{C}$ for both of these sheaves of ideals in $\mathcal{O}_{X'}$ and $\mathcal{O}_X$. Thus $\mathcal{C}$ determines a closed subscheme D of X'. Hence, $x \in D$ if and only if $\mathcal{O}_{X',x}$ is not normal. Since X' has nonsingular normalization, $x \in D$ if and only if x is a singular point of X'. The closed subscheme of X defined by $\mathcal{C}$ is denoted by Δ. Thus, the underlying set of Δ is $f^{-1}(\mathrm{Sing}(X'))$, when $\mathcal{C}$ is considered as a sheaf of ideals in $\mathcal{O}_X$ (see [19]).

The following is an special case of a general result due to D. Rees ([17], also see [12], page 265). We will use its corollary in section 4.

LEMMA 1.3. *Let $(S, \mathfrak{m})$ be an algebro-geometric local ring. If S is reduced, then S is analytically unramified, i.e., its completion is reduced.*

Proof. Consider the inclusion $S \hookrightarrow \prod_1^d S/P_i$, where $(0) = P_1 \cap \cdots \cap P_d$ is the prime decomposition of (0) in S. The last ring is semilocal and its Jacobson radical lies over $\mathfrak{m}$. The completion of this semilocal ring with respect to its Jacobson radical is $\prod_1^d (S/P_i)\widehat{}$ (see [12], Theo. 8.15). By Artin-Rees lemma, the $\mathfrak{m}$-adic topology on S is induced from the topology of $\prod_1^d (S/P_i)$. By flatness of completion, the homomorphism $\widehat{S} \hookrightarrow \prod_1^d (S/P_i)\widehat{}$ is injective. Since each factor $(S/P_i)\widehat{}$ is reduced, $\widehat{S}$ is reduced. $\square$

COROLLARY 1.4. *Let Δ be reduced at some point $x \in X$. Then Δ is analytically unramified at x. Similar statement is true for D.*

Proof. Since $\mathcal{O}_\Delta = \mathcal{O}_X/\mathcal{C}$, $\mathcal{O}_{\Delta,x}$ is an algebro-geometric local ring. Thus by Lemma 1.3, Δ is analytically unramified at x. The proof of the claim for D is similar. $\square$

2. Double points with simple branches

In this section we assume that y is a double point of X' at which X' has two simple branches. Identifying $\widehat{\mathcal{O}}_{Y,y}$ by $R = k[\![u_1, \cdots, u_m]\!]$ and $\widehat{\mathcal{O}}_{X,x_i}$ by $B = k[\![t_1, \cdots, t_r]\!]$, we arrive to a homomorphism $\varphi : R \to B \times B$. Let $\pi_i : B \times B \to B$; $i = 1, 2$, be the projections. If $P_i = \ker(\pi_i \circ \varphi)$, then $\ker\varphi = P_1 \cap P_2$. Thus $\widehat{\mathcal{O}}_{X',y} = R/P_1 \cap P_2$. Since f is finite, R/P_i is r-dimensional, so P_i has height $m - r$. It is known that (see [3], sec. 7; or [16], Lemma 3), in this case, the conductor is $(P_1 + P_2)/P_1 \cap P_2$. If we assume that the conductor has codimension $m - r$ in $R/P_1 \cap P_2$, then $P_1 + P_2$ will be of codimension $(m - r) + (m - r) = 2(m - r)$ in R. We will show that under this assumption, D and Δ are complete intersections at points under consideration.

LEMMA 2.1. *Let P_1 and P_2 be prime ideals in R and let R/P_1, R/P_2 be regular. Then the conductor of $R/P_1 \times R/P_2$ is reduced in $R/P_1 \cap P_2$ if and only if it is reduced in $R/P_1 \times R/P_2$.*

Proof. The integral closure of $R/P_1 \cap P_2$ is $R/P_1 \times R/P_2$. The conductor in this ring is $(P_1+P_2)/P_1 \times (P_1+P_2)/P_2$. Then

$$(R/P_1 \cap P_2)/C \cong R/(P_1+P_2),$$

$$(R/P_1 \times R/P_2)/((P_1+P_2)/P_1 \times (P_1+P_2)/P_2) \cong R/(P_1+P_2) \times R/(P_1+P_2),$$

thus the assertion follows. □

The following is well-known, but we state it for further reference in this paper.

LEMMA 2.2. *With the hypothesis and notation as above, under some change of variables, it can be assumed that* $P_1 = (u_1, \cdots, u_{m-r})$.

Proof. Consider $\varphi_1 : R \to B$ which is surjective by assumption. Let $\varphi_1(f_i) = t_i;\ i = 1, \cdots, r$. Observe that f_i's are of order one, since φ_1 is a homomorphism. We claim that the linear parts of f_i; $i = 1, \cdots, r$, are linearly independent . Since otherwise, $\operatorname{ord}(\lambda_1 f_1 + \cdots + \lambda_r f_r) > 1$ for some λ_i's in k, but then $\varphi_1(\lambda_1 f_1 + \cdots + \lambda_r f_r) = \lambda_1 t_1 + \cdots + \lambda_r t_r$ will be of order greater than one, which is a contradiction. Thus by ([27], Vol II, Ch. VII, Cor.2 of Lemma 2), the change of variables $U_i = f_i;\ i = 1, \cdots, r$, is an isomorphism. Therefore we may assume that $\varphi_1(U_i) = t_i;\ i = 1, \cdots, r$. Now observe that $\varphi_1(u_{r+i}) = g_i(t_1, \cdots, t_r) = \varphi_1(g_i(U_1, \cdots, U_r))$ for $i = 1, \cdots, m-r$, and hence $u_{r+i} - g_i(U_1, \cdots, U_r) \in \ker \varphi_1$. Again, the change of variables $U_{r+i} = u_{r+i} - g_i(U_1, \cdots, U_r)$ is an isomorphism. Thus $(U_{r+1}, \cdots, U_m) \subset \ker \varphi_1$. But since $\ker \varphi_1$ is of height $m-r$, it follows that $P_1 = \ker \varphi_1 = (U_{r+1}, \cdots, U_m)$. □

LEMMA 2.3. *With the assumption as above, let the singular locus of X' be of local dimension $2r-m$ at $y = f(x_1) = f(x_2)$. Then D is a complete intersection at y, and Δ is a complete intersection at x_1 and x_2.*

Proof. By Lemma 2.2, we may assume that $P_1 = (u_1, \cdots, u_{m-r})$. Now we apply Lemma 2.2 for $\varphi_2 : R \to B$. It follows that under similar changes of variables, $P_2 = \ker \varphi_2 = (U_1, \cdots, U_{m-r})$. Changing the variables back to ones used for P_1, we see that $P_2 = (h_1, \cdots, h_{m-r})$ for some $h_i \in R$. Therefore, as an ideal in R, the conductor is $P_1 + P_2 = (u_1, \cdots, u_{m-r}, h_1, \cdots, h_{m-r}) = (u_1, \cdots, u_{m-r}, k_1, \cdots, k_{m-r})$ where $k_i \in k[\![u_{m-r+1}, \cdots, u_m]\!]$. Since the height of this ideal is assumed to be $2(m-r)$, the unmixedness theorem implies that $u_1, \cdots, u_{m-r}, k_1, \cdots, k_{m-r}$ form a regular sequence and hence R/C is a complete intersection.
The claim about Δ follows by the fact that

$$\widehat{\mathcal{O}}_{\Delta, x_i} \cong R/P_1 + P_2;\ i = 1, 2. \qquad \square$$

3. Analytically irreducible double points

Let $f : X \to Y$ be as introduced in section 1. Assume that $y = f(x)$ is an analytically irreducible double point of $X' = f(X)$, i.e., $f^{-1}(f(x)) = \{x\}$, then $\widehat{\mathcal{O}}_{X',y}$ is an integral domain. Let $\varphi : R \to B$ be the homomorphism induced by the local homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, where R and B are the completions of these rings respectively. Assume that y is a double point which is not triple. Then by definition, the B-module $B/\varphi(\mathfrak{m}_R)B$ has length 2. In this section, we will give the "canonical form" of φ. Using this result we will compute the conductor at y. Then assuming that the conductor has codimension $m - r$, we will show that D and Δ are complete intersections at y. This, together with the similar result from section 2, will enable us to compute the depth of the local ring at any double point.

LEMMA 3.1. *Under the assumptions as above, the maximal ideal of $B/\varphi(\mathfrak{m}_R)B$ is principal.*

Proof. If the maximal ideal of $B/\varphi(\mathfrak{m}_R)B$ is generated by more than one elements: $\bar{a}_1, \cdots, \bar{a}_\ell$, using the chain of submodules of $B/\varphi(\mathfrak{m}_R)B$

$$(\bar{1}) \supset (\bar{a}_1, \cdots, \bar{a}_\ell) \supset (\bar{a}_1, \cdots, \bar{a}_{\ell-1}) \supset \ldots \supset (\bar{a}_1) \supset (\bar{0}) \ ,$$

one concludes that $B/\varphi(\mathfrak{m}_R)B$ has length greater than 2, which is a contradiction. $\square$

PROPOSITION 3.2. *Let R and B be as above, and let $B/\varphi(\mathfrak{m}_R)B$ be of length two. There exist automorphisms of R and B such that if we identify these two rings with their images, then φ has the following form :*

i) $\varphi(u_i) = t_i; \quad i = 1, \cdots, r-1,$

ii) $\varphi(u_r) = t_r^2$ *if* $char(k) \neq 2,$

ii′) if $char(k) = 2,$ *then* $\varphi(u_r) \equiv t_r^2$ *mod* $(t_1, \cdots, t_{r-1}, t_r^3),$ *and,* $t_r^2 = \varphi(u_r) + g_0(t_1, \cdots, t_{r-1}, \varphi(u_r))t_r,$ *where g_0 is a non-unit power series in r variables.*

iii) $\varphi(u_{r+i}) = g_i(t_1, \cdots, t_{r-1}, \varphi(u_r))t_r;\ i = 1, \cdots, m-r,$ *where for each i, g_i is a non-unit power series in r variables. Thus, if $char(k) \neq 2$, then $\varphi(u_{r+i}) = g_i(t_1, \cdots, t_{r-1}, t_r^2)t_r;\ i = 1, \cdots, m-r,$ where for each i, g_i is a power series in r variables.*

Proof. Since φ is the completion of a continuous homomorphism (with respect to the adic topologies), φ is continuous. Therefore φ is a substitution map ([27] Vol. II, page 136). Thus, $\varphi(h(u_1, \cdots, u_m)) = h(\varphi(u_1), \cdots, \varphi(u_m))$ for every h in R, and hence it suffices to determine

$\varphi(u_1),\cdots,\varphi(u_m)$. Since the B-module $B/\varphi(\mathfrak{m}_R)B$ has length two, and since φ is a local homomorphism, we may assume that $\varphi(\mathfrak{m}_R)B = (t_1,\cdots,t_{r-1},t_r^2)$. Thus by some linear change of variables in R, it can be assumed that

$$\begin{aligned} \varphi(u_i) &= t_i \ ; \ i=1,\cdots,r-1, \\ \varphi(u_r) &\equiv t_r^2 \ \mathrm{mod}(t_1,\cdots,t_{r-1},t_r^3). \end{aligned} \tag{$\star$}$$

In particular we get $i)$ and the first part of $ii')$. Let $A_0 = k[\![u_1,\cdots,u_r]\!]$ and let $\mathfrak{m}_0$ be the maximal ideal of A_0. We first claim that B is a finite A_0-module. Observe that

$$\varphi(u_r) \equiv (1+h(t_r))t_r^2 \ \mathrm{mod}(\varphi(\mathfrak{m}_0)B),$$

where h is a non-unit power series in t_r. Thus $t_r^2 \in \varphi(\mathfrak{m}_0)B$, i.e., $\varphi(\mathfrak{m}_0)B$ is primary for $\mathfrak{m}_B$. Therefore B is a finite A_0-module (see [27] Vol. II, page 211). While $B/\varphi(\mathfrak{m}_0)B$ is now generated by $\bar{1}$ and $\bar{t}_r$, by Nakayama's lemma (or equivalently, by [4], Theo. 30.6), B is generated by 1 and t_r as an A_0-module. Therefore, $t_r^2 = \varphi(g)t_r + \varphi(h)$ for some non-unit power series $g, h \in A_0$, which gives a relation of integral dependence of t_r over A_0. Now we claim that we may assume that $h = u_r$. Observe that by $(\star)$,

$$\begin{aligned} \varphi(u_r) = (1+a(t_1,\cdots,t_{r-1}))t_r^2 \ &+ b(t_1,\cdots,t_{r-1})t_r \\ &+ c(t_1,\cdots,t_{r-1}) + \ d(t_1,\cdots,t_r)t_r^3, \end{aligned}$$

where a, b, c and d are power series and a is non-unit. By the change of variable $u_r - c(u_1,\cdots,u_{r-1})$ to u_r, we may assume that $c = 0$. By changing $u_r/(1+a(u_1,\cdots,u_{r-1}))$ to u_r, it can be assumed that $a = 0$. Thus we have

$$\varphi(u_r) = t_r^2 + b(t_1,\cdots,t_{r-1})t_r + d(t_1,\cdots,t_r)t_r^3, \tag{$\star\star$}$$

where b is non-unit, because otherwise, $t_r \in \varphi(\mathfrak{m}_0)B \subset \varphi(\mathfrak{m}_R)B$, so that $\varphi(\mathfrak{m}_R)B = \mathfrak{m}_B$ which contradicts Lemma 3.1. Now consider the relation $\varphi(h) = t_r^2 - \varphi(g)t_r$, which may be written as

$$h(t_1,\cdots,t_{r-1},\varphi(u_r)) = t_r^2 - g(t_1,\cdots,t_{r-1},\varphi(u_r))t_r.$$

Substituting from $(\star\star)$, since b is non-unit, the coefficient of t_r^2 on the right hand side of the above equality is a unit. Since b^2 will also be a non-unit, in order to get a unit coefficient for t_r^2 on the left hand side of the above equality, the coefficient of u_r in the linear part of $h(u_1,\cdots,u_r)$ must be nonzero. Consequently, we may change h to u_r to arrive to

$$\varphi(u_r) = t_r^2 + \varphi(g_0)t_r = t_r^2 + g_0(t_1,\cdots,t_{r-1},\varphi(u_r))t_r.$$

This settles $ii')$. If $\mathrm{char}(k) \neq 2$, then by "completing the square" above, i.e., by changing $t_r - \varphi(g_0)/2$ into t_r, $ii)$ follows. Now since B is generated by 1 and t_r as an A_0-module, we get

$$\varphi(u_{r+i}) = \varphi(f_i(u_1, \cdots, u_r)) + \varphi(g_i(u_1, \cdots, u_r))t_r \ ; \ i = 1, \cdots, m-r,$$

for non-unit power series f_i and g_i in r variables. Indeed, since completion is an exact functor, by ([21], Lemma 2.7), B is free A_0-module, so that f_i's and g_i's are unique. Replacing $u_{r+i} - f_i(u_1, \cdots, u_r)$ by u_{r+i} for $i = 1, \cdots, m-r$, we may assume that

$$\varphi(u_{r+i}) = \varphi(g_i(u_1, \cdots, u_r))t_r = g_i(t_1, \cdots, t_{r-1}, \varphi(u_r))t_r,$$

concluding $iii)$. □

Let $A = \varphi(R)$. For simplicity, we identify A_0 with its image $\varphi(A_0) \subset B$. Thus $u_i = t_i$ for $i = 1, \cdots, r-1$, and $u_r = t_r^2$ if $\mathrm{char}(k) \neq 2$ (resp. $u_r \equiv t_r^2 \bmod (t_1, \cdots, t_{r-1}, t_r^3)$ in general). Hence A_0 is identified with the subring $k[\![t_1, \cdots, t_{r-1}, \varphi(u_r)]\!] \subset B$. Observe that regardless of the characteristic, we have $B = A_0 \oplus A_0 t_r$. Let C be the conductor of B in $A = \varphi(R)$. It is the annihilator of B/A as an A-module which is also an ideal in B. Since $g_i(t_1, \cdots, t_{r-1}, \varphi(u_r))$ and $g_i(t_1, \cdots, t_{r-1}, \varphi(u_r))t_r$ belong to $A = \varphi(R)$, we see that $g_i \in C$.

PROPOSITION 3.3. *Let the hypotheses and notation be as in Proposition 3.2. Then, the conductor C as an ideal in B is generated by $g_1, \cdots, g_{m-r}$ and as an ideal in R is generated by $g_1, \cdots, g_{m-r}, g_1 t_r, \cdots, g_{m-r} t_r$.*

Proof. Let $f \in C$ and let $b \in B$, then

$$f = f_0(t_1, \cdots, t_{r-1}, u_r) + f_1(t_1, \cdots, t_{r-1}, u_r)t_r \ ,$$
$$b = p(t_1, \cdots, t_{r-1}, u_r) + q(t_1, \cdots, t_{r-1}, u_r)t_r \ ,$$

for some power series f_0, f_1, p and q in r variables. For every $b \in B$ we have $fb \in \varphi(R)$. In particular, for every $p, q \in A_0$ we have:

$$fp = f_0 p + f_1 p t_r \in A, \quad f q t_r = f_1 q t_r^2 + f_0 q t_r \in A \ . \qquad (\star\star\star)$$

Observe that $\varphi(R) = A_0 \oplus \Gamma t_r$ where $\Gamma = (g_1, \cdots, g_{m-r})A_0$. As an element of the direct sum $A_0 \oplus \Gamma t_r$, both fp and fqt_r have unique representations. Hence by $(\star\star\star)$, $f_1 p \in \Gamma$ for every $p \in A_0$. Thus $f_1 \in \Gamma$. Now if $\mathrm{char}(k) \neq 2$, $(\star\star\star)$ implies that $f_0 q t_r \in \Gamma t_r$ and hence $f_0 q \in \Gamma$ for every $q \in A_0$. It follows that $f_0 \in \Gamma$. If $\mathrm{char}(k) = 2$, we substitute from Prop. 3.2 $ii')$ into $(\star\star\star)$ to obtain:

$$f q t_r = f_1 q \varphi(u_r) + (f_0 + g_0 f_1) q t_r \ ,$$

so that $f_0 + g_0 f_1 \in \Gamma$. Since $f_1 \in \Gamma$, it follows again that $f_0 \in \Gamma$. Therefore $C = (g_1, \cdots, g_{m-r})B$. Since $B = A + At_r$, we get

$$C = (g_1, \cdots, g_{m-r}, g_1 t_r, \cdots, g_{m-r} t_r)A$$

as an ideal in A. □

COROLLARY 3.4. *With the notation as in Prop. 3.3 and its proof:*
i) $C = \Gamma \oplus \Gamma t_r$,
ii) $B/C \cong A_0/\Gamma \oplus (A_0/\Gamma)t_r$,
iii) $A/C \cong A_0/\Gamma$, *and*
iv) $B/A \cong (A_0/\Gamma)t_r$.
In particular, B/A is a free A/C-module of rank 1.

Proof. i) Follows from Prop. 3.3 and the fact B is a free A_0-module generated by $1, t_r$.
ii) Follows from this latter fact and *i*).
iii) This is a direct consequence of Prop. 3.3.
iv) This follows from *ii*) and *iii*).
The statement about B/A follows from *iii*). □

COROLLARY 3.5. *With the assumption as in Prop. 3.3, if furthermore Sing(X') is of local dimension $2r - m$ at y, then Δ is a complete intersection at x.*

Proof. Observe that D and Δ have the same dimensions as $\mathrm{Sing}(X')$. Using the same notation as in Prop. 3.3, Δ is locally given by the ideal $C = (g_1, \cdots, g_{m-r}) \subset B$. By assumption, C has height $r - (2r - m) = m - r$. Thus C is unmixed, and since B is Cohen-Macaulay, $g_1, \cdots, g_{m-r}$ form a regular sequence. Consequently B/C is a complete intersection. □

REMARK 3.6. By the conductor, one usually means $(\varphi(R) : B)$. Here the inverse image of the this ideal in R will also be called the conductor.

LEMMA 3.7. *Let $\varphi : T \to S$ be any surjective homomorphism of commutative rings. Let $I = (s_1, \cdots, s_m)$ be an ideal in S. Assume that $\varphi(f_i) = s_i;\ i = 1, \cdots, m$, for some elements $f_i \in T$. Then*

$$\varphi^{-1}(I) = ker\varphi + (f_1, \cdots, f_m).$$

In particular if ker $\varphi \subset (f_1, \cdots, f_m)$, then $\varphi^{-1}(I) = (f_1, \cdots, f_m)$.

Proof. Let $\bar{\varphi} : T/\ker\varphi \overset{\cong}{\to} S$ be the isomorphism induced by φ. Thus $\bar{\varphi}^{-1}(I) = (\bar{f}_1, \cdots, \bar{f}_m)$, and hence $\varphi^{-1}(I) = (f_1, \cdots, f_m) + \ker\varphi$. □

LEMMA 3.8. *Let $\varphi : R \to B$ be the homomorphism defined in Prop. 3.2, then ker $\varphi \subset (u_{r+1}, \cdots, u_m, g_1, \cdots, g_{m-r})$.*

Proof. Let $\tilde{\varphi} : R[t_r] \to B$ be the homomorphism extending φ by the identity on t_r. Then

$$\ker\tilde{\varphi} = (u_{r+i} - g_i(u_1, \cdots, u_r)t_r;\ \ i = 1, \cdots, m-r,\ u_r - \varphi(u_r)),$$

and

$$\ker\varphi = \ker\tilde{\varphi} \cap R \subset (u_{r+i}, g_i(u_1, \cdots, u_r);\ \ i = 1, \cdots, m-r,\ u_r - \varphi(u_r)) \cap R$$

$$= (u_{r+i},\ g_i(u_1, \cdots, u_r);\ \ i = 1, \cdots, m-r). \qquad \square$$

Although the following is a consequence of Cor. 3.4 *iii*), in order to prove it, we prefer to use the generating set of the conductor as an ideal in R.

COROLLARY 3.9. *With the notation as above,*

$$\varphi^{-1}(C) = (u_{r+i},\ g_i;\ i = 1, \cdots, m-r).$$

In particular A/C is a complete intersection. Therefore D is a complete intersection at analytically irreducible double points of $X' = f(X)$.

Proof. The first claim follows from Prop. 3.3, Lemmas 3.7 and 3.8. Since $g_1, \cdots, g_{m-r}$ form a regular sequence in $k[\![u_1, \cdots, u_r]\!]$, it is clear that $g_1, \cdots g_{m-r}, u_{r+1}, \cdots, u_m$ will also form a regular sequence in R. Therefore $R/\varphi^{-1}(C) \cong A/C$ is a complete intersection. □

LEMMA 3.10. *Let $\bar{t}_r$ be the class of t_r in B/A. Then the conductor as an ideal in A, is equal to $Ann_A(\bar{t}_r)$.*

Proof. Since B is generated by 1, t_r, as an A-module, B/A is generated by $\bar{t}_r$ as an A-module. Thus by definition,

$$C = \mathrm{Ann}_A(B/A) = \mathrm{Ann}_A(\bar{t}_r). \qquad \square$$

REMARK 3.11. In Cor. 3.9, we have seen that A/C is a complete intersection, hence it is Cohen-Macaulay. A similar result is not true for analytically irreducible triple points, even with the hypothesis that

f is an strongly generic projection. For example, it is shown in ([21], Theo. 4.12) that

$$\mathrm{depth} A/C = 3r - 2m + 1 \ ,$$

at an analytically irreducible triple point where:

$$r = 2n + 2 \ , \quad m = 3n + 2 \ ,$$

so that the above number is 3. Since

$$\dim A/C = 2r - m,$$

we see that A/C is Cohen-Macaulay if and only if

$$3r - 2m + 1 = 2r - m,$$

i.e., if and only if $m = r+1$. Consequently, A/C is not Cohen-Macaulay at y if $m > r + 1$. For B/C there is a similar situation.

PROPOSITION 3.12. *Let $f : X \to Y$ be a finite morphism as specified in section 1. Let $y \in X' = f(X)$ be a point of multiplicity 2. Assume that Sing(X') is of local dimension $2r - m$ at y. Then*

$$\textit{depth}\ \mathcal{O}_{X',y} = 2r - m + 1.$$

Proof. If $m = r + 1$, then X' is a hypersurface and depth $\mathcal{O}_{X',y} = r$. Thus we work out the case $m > r+1$. First assume that y is analytically irreducible. Let $A = \mathcal{O}_{X',y}, B = \mathcal{O}_{X,x}$. Thus B is the integral closure of A. Let C be the conductor of B in A. By Cor. 3.4, B/A is a free A/C-module, thus it is a Cohen-Macaulay A/C-module. Therefore B/A is a Cohen-Macaulay A-module, so that

$$\mathrm{depth}_A(B/A) = \dim(B/A) = 2r - m.$$

Now consider the exact sequence of A-modules

$$0 \to A \to B \to B/A \to 0.$$

Since B is Cohen-Macaulay and finite over A,

$$\mathrm{depth}_A B = \mathrm{depth}\ B = r.$$

By assumption A has R_1 property (regular in codimension 1), thus A is not Cohen-Macaulay. Because otherwise A will be integrally closed which is a contradiction. Therefore

$$\mathrm{depth}\ A < \mathrm{depth}_A B.$$

By the behavior of depth on exact sequences (see [2], Lemma 1.4),

$$\operatorname{depth}_A(B/A) = \operatorname{depth} A - 1,$$

and hence the assertion follows.

Now assume that y is a double point at which X' has two simple branches. Using the same notation as in Lemma 2.1, let $A = R/P_1 \cap P_2$. Then $B = R/P_1 \times R/P_2$ is the integral closure of A in its total ring of quotients. If C is the conductor, by the proof of Lemma 2.1,

$$A/C \cong R/(P_1 + P_2),$$

and,

$$B/C \cong R/(P_1 + P_2) \times R/(P_1 + P_2).$$

Consequently, the exact sequence of A/C-modules

$$0 \to A/C \to B/C \to B/A \to 0,$$

splits. Hence B/A is a projective A/C-module. While A/C is local, B/A is a free A/C-module. Now as the previous case, using the depth relation for a similar exact sequence, the assertion follows. $\square$

Let $X \subset \mathbb{P}^N$ be a projective smooth variety with no trisecant lines. The projection of X from any point outside X into $\mathbb{P}^{N-1}$ has no triple point. The study of these varieties is a classical problem in algebraic geometry. The case of space curves goes back to G. Castelnuovo. The general problem is still far from being settled. The surfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$ with no trisecant lines are characterized and the list of surfaces in $\mathbb{P}^6$ with this property has been conjectured by S. Di Rocco and K. Ranestad (see [28]). However, it is known that if $\pi : X \to \mathbb{P}^m$ is a generic projection, for $\frac{3r}{2} \leq m \leq 2r$, π has no triple point ([18], sec. 12). For finite morphisms with a similar property we have the following.

THEOREM 3.13. *Let $f : X \to Y$ be a finite morphism specified as in section 1. Assume that f has no triple point. If the singular locus of $X' = f(X)$ is of dimension $2r - m$, then D and Δ are locally complete intersections.*

Proof. This follows by Lemma 2.3, Cor. 3.5 and Cor. 3.9. $\square$

4. Weak Normality

In this section we will keep the notations used in the previous sections. We will assume that $\mathrm{char}(k) \neq 2$. We will check seminormality and weak normality of the varieties and schemes introduced in the earlier sections at the double points. In particular, we will prove that when the conductor, as a subscheme of X, is reduced, then $X' = f(X)$ is WN at double points. This implies the global result that when Δ is reduced, X' is WN provided that it has no triple point. Assuming that $\mathrm{Sing}(X')$ has the expected dimension, this result is strengthened.

LEMMA 4.1. *Let y be a double point of X' at which X' has two simple branches. If $\mathcal{O}_{X',y}$ is SN, then it is also WN.*

Proof. By assumption $\widehat{\mathcal{O}}_{X',y} \cong R/P_1 \cap P_2$, where R/P_1 and R/P_2 are regular. Since $\mathcal{O}_{X',y}$ is SN, $\widehat{\mathcal{O}}_{X',y}$ is SN. To see $\mathcal{O}_{X',y}$ is WN, it is sufficient to show that $\widehat{\mathcal{O}}_{X',y}$ is WN ([11], II,3). Let $A = R/P_1 \cap P_2$, then the integral closure of A is $B = R/P_1 \times R/P_2$ ([4], Ch. V, Prop. 9). So by Prop. 1.2, we need to check that for $b \in B$, if pb, $b^p \in A$, for some prime integer p then $b \in A$. Let $b = (\overline{\alpha}, \overline{\beta}) \in B$, then $pb = (\overline{p\alpha}, \overline{p\beta}) = (\overline{\gamma}, \overline{\gamma})$ for some $\gamma \in R$. If $p \neq \mathrm{char} k$, since $k \subset \widehat{\mathcal{O}}_{X',y}$, $\overline{\frac{1}{p}\gamma} \in A$ which is mapped to $b = (\overline{\frac{1}{p}\gamma}, \overline{\frac{1}{p}\gamma})$. If $p = \mathrm{char}\ k$, and $b^p \in A$, then $(\overline{\alpha^p}, \overline{\beta^p}) = (\overline{\delta}, \overline{\delta})$ for some $\delta \in R$, thus $\alpha^p - \delta \in P_1, \beta^p - \delta \in P_2$ and hence $\alpha^p - \beta^p = (\alpha - \beta)^p$ is in $P_1 + P_2$. Since A is SN, $P_1 + P_2$ is radical in A, therefore $\alpha - \beta \in P_1 + P_2$. Let $\alpha - \beta = g + h$ with $g \in P_1$, $h \in P_2$. Thus the class of $\alpha - g = \beta + h$ in A maps to $(\overline{\alpha}, \overline{\beta})$, i.e., $b \in A$. □

LEMMA 4.2. *Let $y \in X'$, be as in Lemma 4.1. Furthermore assume that D is reduced at y. Then $\mathcal{O}_{X',y}$ is SN.*

Proof. Using the same notation as above, $\widehat{\mathcal{O}}_{X',y} \cong R/P_1 \cap P_2$ and its integral closure is $R/P_1 \times R/P_2$. By Cor. 1.4, D is analytically unramified at y. This means that the conductor of $R/P_1 \cap P_2$ in $R/P_1 \times R/P_2$ is a radical ideal in $R/P_1 \cap P_2$, because the completion of the conductor of two rings is the conductor of the completion of these rings ([27], Vol. II, Ch. VIII, Cor. 8 to Theo. 11). This implies seminormality of $\widehat{\mathcal{O}}_{X',y}$ and hence that of $\mathcal{O}_{X',y}$ by ([16],Theo. 2.3(d)). □

COROLLARY 4.3. *Under the assumptions as in Lemma 4.2, $\mathcal{O}_{X',y}$ is WN.*

Proof. This follows by Lemmas 4.1 and 4.2. □

PROPOSITION 4.4. *Let $y = f(x)$ be an analytically irreducible double point of X', and let $B = \mathcal{O}_{X,x},\ A = \mathcal{O}_{X',y},\ C = (A : B)$. Assume that Δ is reduced and char $k \neq 2$. Then A/C is SN in B/C, and $\mathcal{O}_{X',y}$ is SN.*

Proof. As indicated in the proof of Lemma 4.2, $\widehat{C} = (A : B) \otimes \widehat{A} = (\widehat{A} : \widehat{B})$. Since seminormality descends under completion, we may assume that $B = \widehat{\mathcal{O}}_{X,x}, A = \widehat{\mathcal{O}}_{X',y}$ and $C = (A : B)$. Recall that by Cor. 3.4 *iii*), $B/C = A/C \oplus (A/C)t_r$. Let $b = \alpha + \beta t_r \in B/C$ with $\alpha, \beta \in A/C$ and let $b^2, b^3 \in A/C$. Then $b^2 = (\alpha^2 + \beta^2 t_r^2) + (2\alpha\beta)t_r \in A/C$. Since the sum $A/C \oplus (A/C)t_r$ is direct, $2\alpha\beta t_r = 0$, and since char $k \neq 2, \alpha\beta t_r = 0$. On the other hand, $b^3 = (\alpha^3 + 3\alpha\beta^2 t_r^2) + 3\alpha^2\beta t_r + \beta^3 t_r^3 \in A/C$. Since $\alpha^2\beta t_r = \alpha(\alpha\beta t_r) = 0$ again, $\beta^3 t_r^3 \in (A/C) \cap (A/C)t_r = 0$, thus $\beta^3 t_r^3 = 0$ in B/C. Since B/C is reduced, $\beta t_r = 0$, and hence $b = \alpha \in A/C$. Therefore A/C is SN in B/C. Now by ([6], Lemma 2.5 (VI)), A is SN in B. In other words, $\mathcal{O}_{X',y}$ is SN. □

PROPOSITION 4.5. *With the assumption of Prop. 4.4, $\mathcal{O}_{X',y}$ is WN.*

Proof. By Prop. 4.4, $\mathcal{O}_{X',y}$ is SN. By Prop. 1.2, we need to verify that for all $b \in B$, if $pb, b^p \in A$, for some prime p, then $b \in A$. In our case B is generated by 1, t_r and by Lemma 3.10, C is the annihilator of $\bar{t}_r$ in B/A as an ideal of A. Assume that $b = a + a't_r \in B, pb = pa + pa't_r \in A$, so $pa't_r \in A$ and hence $pa' \in C$. Now assume that $b^p = (a + a't_r)^p = a^p + pa'c + (a't_r)^p \in A$, where $c \in B$. Then $pa'c \in A$ and $(a't_r)^p \in A$. For $p > 2$, $p - 1$ is even and ${a'}^p t_r^p \in A$, hence ${a'}^p t_r^{p-1} \in C$, i.e., $a't_r$ is nilpotent in B/C. Since $a't_r \in B$ and B/C is reduced, $a't_r \in C \subset A$. Therefore $b = a + a't_r \in A$. If $p = 2$, then $2a' \in C$, and since char $k \neq 2, a' \in C$, thus $a't_r \in C \subset A$, and hence $b \in A$. □

COROLLARY 4.6. *Under the assumptions of Prop. 4.4, A/C is WN in B/C.*

Proof. (see [24], Prop. 3). □

The following result is a partial generalization of ([20], Prop. 4.1).

PROPOSITION 4.7. *Under the assumptions of Prop. 4.4, if X' is SN at y and Δ is SN at x, then D is SN at y.*

Proof. Let $A = \mathcal{O}_{X',y},\ B = \mathcal{O}_{X,x},\ C = (A : B)$. Since A is SN in B ([6], Prop. 2.5(VI)), A/C is SN in B/C. By assumption, B/C is SN in $\bar{(B/C)}$. Thus by transitivity of seminormality, A/C is SN in $\bar{(B/C)}$.

Since $(A/C)^{\bar{}} \subset (B/C)^{\bar{}}$ (see [20], the proof of Prop. 4.1), by ([6], Prop. 1.5(b)), A/C is SN. □

In the next two results we will consider a finite morphism $f : X \to Y$ as specified in sec. 1. We will assume that f has no triple point.

The following generalizes ([1], Theo. 2.7), ([10], Prop. 3.5) and ([19], Prop. 4.4).

THEOREM 4.8. *With the notation as above, if Δ is a reduced scheme, then $X' = f(X)$ is WN.*

Proof. This follows from Cor. 4.3 and Prop. 4.5. □

In the following, by a point, we mean a scheme-theoretic point, i.e., it is not necessarily a closed point.

Theorem 4.8 can be strengthened in the following sense.

THEOREM 4.9. *With the notation as above, let $r+1 \leq m \leq 2r-1$. Assume that :*

i) Sing(X') has dimension $2r-m$,

ii) X' has no triple point,

iii) On each irreducible component of Sing(X') there is at least one point where X' has two simple branches, on which D is reduced.

Then X' is WN.

Proof. We may restrict the problem to the case when $X = \mathrm{Spec}B$, $X' = \mathrm{Spec}A$ are affine varieties. Let C be the conductor. Let $P \subset A$ be a prime ideal containing C such that its image is of height h in A/C. It corresponds to a point $\xi \in D$. We first show that $\mathrm{depth}\mathcal{O}_{X',\xi} = h+1$. For $m = r+1$, $\mathcal{O}_{X',\xi}$ ia a Cohen-Macaulay ring, hence,

$$\text{depth } \mathcal{O}_{X',\xi} = \dim \mathcal{O}_{X',\xi} = h+1.$$

Assume that $m > r+1$. As it was seen in the proof of Prop. 3.12, B/A is a Cohen-Macaulay A-module. Thus $(B/A)_P$ is a Cohen-Macaulay A_P-module. Now apply the method of the proof of Prop. 3.12 to the exact sequence

$$0 \to A_P \to B_P \to (B/A)_P \to 0.$$

Since $m > r+1$, by i), A is regular in codimension one, thus A_P is regular in codimension one. Since $C \subset P$, A_P is not normal. Consequently, A_P is not Cohen-Macaulay. Thus as A_P-modules,

$$\mathrm{depth}(A_P) < \mathrm{depth}(B_P).$$

Hence, by the behavior of depth on exact sequences,

$$\text{depth } \mathcal{O}_{X',\xi} = \text{depth}(B/A)_P + 1 = h + 1.$$

If y is the double point of X' on which D is reduced, then by Lemmas 4.2 and 4.3, $\mathcal{O}_{X',\xi}$ is WN. Thus for $h = 0$, $\mathcal{O}_{X',\xi}$ is WN. For $h > 1$, depth$\mathcal{O}_{X',\xi} \geq 2$. Thus by ([25], Prop. 2.10), X' is SN. Therefore, by Lemma 4.1 and Prop. 4.5, X' is WN. □

For a local ring $(A, \mathfrak{m})$, let $\text{Spex}(A) = \text{Spec}(A) \setminus \{\mathfrak{m}\}$. A result of M. Vitulli together with Theo. 4.9, yield the following necessary and sufficient version of Hartshorne's depth-connectivity result (see [7], Prop. 2.1).

PROPOSITION 4.10. *With the assumption of Theo. 4.9, let ξ be a singular point of X'. Then Spex($\widehat{\mathcal{O}}_{X',\xi}$) is connected if and only if ξ is not a generic point of Sing(X').*

Proof. By Theo. 4.9, the complete local ring $\widehat{\mathcal{O}}_{X',\xi}$ is SN. Let ξ be a singular point of X'. By assumption $i)$ of Theo. 4.9, $\dim(\widehat{\mathcal{O}}_{X',\xi}) \geq 2$. By the proof of Theo. 4.9, depth $(\widehat{\mathcal{O}}_{X',\xi}) \geq 2$ if and only if ξ is not a generic point of Sing(X'). Therefore by ([23], Cor. 3.4), to prove the claim, it is sufficient to show that $\widehat{\mathcal{O}}_{X',\xi}$ has rational normalization, i.e., its residue field is equal to the residue field of its normalization. But this is immediate since $\widehat{\mathcal{O}}_{X',\xi}$ has finite normalization and k is algebraically closed. □

5. The case of affine spaces

In this section we consider "the affine model" of the analytically irreducible double points, studied in section 3. For simplicity, we will assume that char(k) $\neq 2$. More precisely, assume that $R = k[u_1, \cdots, u_m]$, $B = k[t_1, \cdots, t_r]$ and let $g_1(t_1, \cdots, t_{r-1}, t_r^2), \cdots, g_{m-r}(t_1, \cdots, t_{r-1}, t_r^2)$ be polynomials in B which form a regular sequence. Let $\varphi : R \to B$ be defined similar to Prop. 3.2, namely,

$\varphi(u_i) = t_i; \qquad i = 1, \cdots, r-1,$
$\varphi(u_r) = t_r^2,$
$\varphi(u_{r+i}) = g_i t_r; \quad i = 1, \cdots, m-r.$

Let $X = \mathbb{A}_k^r, Y = \mathbb{A}_k^m$ and let $f : X \to Y$ be the morphism corresponding to φ. It follows that f is a finite morphism which is birational onto $X' = f(X)$ and X' has a double point at the origin which is not a triple point.

We give a finite presentation for B as an R-module. This in particular gives the Fitting ideals of B as an R-module. The Fitting ideals are not usually radical (see [13], discussion after Prop. 1.5). It turns out that in this situation, the 0-th Fitting ideal is a prime ideal and indeed it is the defining ideal of X'. While in general, the first Fitting ideal is contained in the conductor of B in $A = \varphi(R)$ (see [13], Theo. 3.4), in this case equality holds.

PROPOSITION 5.1. *The sequence of R-moules*

$$R^{2(m-r)} \to R^2 \to B \to 0,$$

where the first map is defined by the matrix

$$M = \begin{pmatrix} u_{r+1} & g_1 \\ u_r g_1 & u_{r+1} \\ u_{r+2} & g_2 \\ u_r g_2 & u_{r+2} \\ \vdots & \vdots \\ u_m & g_{m-r} \\ u_r g_{m-r} & u_m \end{pmatrix},$$

and the second map is defined as

$$(a, a') \mapsto a.1 + a'.t_r,$$

is exact, i.e., it gives a finite presentation for B as an R-module.

Proof. As it was seen in the proof of Prop. 3.2, B is generated by 1 and t_r as an R-module. So we only need to show that if $a.1 + a't_r = 0$, then (a, a') is generated by the rows of the matrix M. We subtract certain multiples of the rows of M from (a, a') to arrive to zero. Let

$$s = \text{Max}\{\text{degree of } u_{r+1} \text{ in } a \ , \ \ \text{degree of } u_{r+1} \text{ in } a'\}.$$

Multiplying (u_{r+1}, g_1) by an appropriate factor and subtracting it from (a, a'), we can reduce the degree of u_{r+1} in a by at least 1, and using $(u_r g_1, u_{r+1})$ in a similar manner, we can reduce the degree of u_{r+1} in a' by at least 1. Thus, we can reduce s by at least one. Repeating this process, we arrive to the case where a, a' are independent of u_{r+1}. The same argument applies to u_{r+2}, using the third and fourth rows of M. Continuing this for remaining rows, we arrive to an element (a, a') where a and a' are independent of $u_{r+1}, \cdots, u_m$. Since B is a free module over $k[u_1, \cdots, u_r]$, $a.1 + a't_r = 0$ implies that $a = a' = 0$. $\square$

THEOREM 5.2. *Let F_0 be the 0-th Fitting ideal of B as an R-module. The defining ideal of X' is equal to F_0. In other words, the defining ideal of X' is generated by the maximal minors of M.*

Proof. It is known that $X' = V(F_0)$. On the other hand, ker φ is the defining ideal of the closure of $X' = f(X)$ which is closed as f is a finite morphism. Therefore rad$(F_0) =$ ker φ. In particular, $F_0 \subset$ ker φ. Let g belong to ker φ. Then $(g, 0)$ is a syzygy of $(1, t_r)$. By Prop. 5.1, the module of syzygies of $(1, t_r)$ is generated by the rows of M. Therefore there exist $a_1, \cdots, a_{2(m-r)}$ in R such that

$$g = a_1 u_{r+1} + a_2 u_r g_1 + \cdots + a_{2(m-r)} u_r g_{m-r},$$

and,

$$0 = a_1 g_1 + a_2 u_{r+1} + \cdots + a_{2(m-r)} u_m.$$

Since $g_1, \cdots, g_{m-r}$ form a regular sequence, and they are polynomials in $u_1, \cdots, u_r$, the sequence $g_1, u_{r+1}, g_2, u_{r+2}, \cdots, g_{m-r}, u_m$ is also regular, i.e., the second column of M form a regular sequence. By a well known result in commutative algebra, the module of syzygies of a regular sequence is generated by the so called "trivial syzygies". These are, by definition, syzygies of the form

$$(u_{r+1}, -g_1, 0, \cdots, 0),\ (g_2, 0, -g_1, 0, \cdots, 0),\ \cdots,\ (0, \cdots, 0, u_m, -g_{m-r})\ .$$

Taking $(a_1, \cdots, a_{2(m-r)})$ to be equal to the first syzygy, we see that

$$g = (u_{r+1})^2 - u_r g_1^2,$$

which is a maximal minor of M. Similarly, the other trivial syzygies give rise to maximal minors of M. Thus g is generated by the maximal minors of M, i.e., $g \in F_0(M)$. □

COROLLARY 5.3. *With the notation as above, the 1-st Fitting ideal of B as an R-module is equal to the conductor of B in A (as an ideal in R, see Remark 3.6).*

Proof. Similar to the proof of Cor. 3.9, the conductor is generated by $u_{r+1}, \cdots, u_m,\ g_1, \cdots, g_{m-r}$ as an ideal in R. By Prop. 5.1, F_1 is generated with the entries of M which gives the same generators as above. □

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