Math 5335 Fall 2004
Comments about Problem 5.39

We're supposed to study the compositions $\mathcal{R}\circ\mathcal{G}$ and $\mathcal{G}\circ\mathcal{R}$, where $\mathcal{R}$ is the $\pi/4$ rotation about $(0,0)$ and $\mathcal{G}$ is the glide reflection formed as the composition of reflection across the line $y = x$ with translation by $(2,2)$. Specifically, we're supposed to determine what they are – presumably reflections or glide reflections – and in the case of a glide reflection, we're supposed to find both the mirror line and the glide.

Now, $\mathcal{R}$ is just defined by the $\pi/4$ rotation matrix, so that all of the entries are $\pm\sqrt{2}/2$. Reflection across the line $y = x$ is defined by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and we obtain $\mathcal{G}$ by composing this reflection with translation by $(2,2)$. Therefore

$$[\mathcal{G}(X)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}[X] + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

We begin, of course, by finding the two compositions. And then we look for fixed points to determine whether they are reflections or glide reflections – or conceivably one of each. Now, there's one small glitch that happens already when we're looking for fixed points: the equations are slightly messy. Here's a sample of the kind of thing you might find:

$$(\sqrt{2} - 2)x_1 + \sqrt{2}x_2 = -4$$
$$\sqrt{2}x_1 - (\sqrt{2} + 2)x_2 = -4.$$

If you multiply the second equation by $\sqrt{2}$ and then by $\sqrt{2} - 2$, you'll have an equation whose left side is very closely comparable to the left side of the first equation. Just do the algebra carefully when working with the square roots.

In the case where the composition is a glide reflection, we need to find the glide vector and the mirror line of the reflection part. Thus, let's suppose that $[\mathcal{G}\circ\mathcal{R}(X)] = A[X] + [S]$, where A is a reflection matrix, and $[S]$ is the shift vector. As I said in class, I claim that we can do this by writing the shift vector as a sum $[S] = [P] + [Q]$, where $[Q]$ is a direction vector of the mirror line of the reflection defined by the matrix A and $[P]$ is orthogonal to $[P]$. And then the reflection will be defined as $\mathcal{M}[X] = A[X] + [P]$, while $[Q]$ will be the glide vector. And, by the way, in this formula, the matrix A is the same as in the formula for $\mathcal{G}\circ\mathcal{R}(X)$.

Now, why does this work? Well, a basic fact (which we haven't discussed very much) is that if $\mathcal{M}[X] = A[X] + [P]$ is a reflection, then $[P]$ is perpendicular to the mirror line of $\mathcal{M}$. One way to see this involves looking at equation 5.10 in §5.3. In that equation, we take $(\cos\varphi, \sin\varphi)$ as the direction vector of the mirror line, so that $(\sin\varphi, -\cos\varphi)$ is the unit normal vector, ***i.e.*** perpendicular to the mirror line, and the equation of the mirror line is $\langle(\sin\varphi, -\cos\varphi), X\rangle = c$. Then equation 5.10 says that

$$\mathcal{M}[X] = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{bmatrix} + 2c\begin{bmatrix} \sin\varphi \\ -\cos\varphi \end{bmatrix},$$

and the shift vector is clearly exhibited as a scalar multiple of the normal vector.

To implement this, we find a direction vector of the mirror line of the reflection of our matrix A, (found when we calculated $\mathcal{G}\circ\mathcal{R}$). We call this vector U, and we take V to be a vector perpendicular to U. Contrary to what I said in class, ***we don't have to take them to be unit vectors***. For instance if $[U] = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$, then we can take $[V] = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$. {These may not be the actual vectors that you'll obtain.} We need to solve the equation $[S] = a\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} + b\begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$ to find suitable coefficients. In a direct approach, the resulting system of equations is about as messy as the ones that we encountered before – but it can be solved. Alternatively, you may find some sort of trial and error process to be easier if only you could avoid the "comparing apples and oranges" situation that we have with the present U and V. For instance, if you'd like to have the square roots at the bottom in both vectors, then try multiplying $[V]$ by $1+\sqrt{2}$. Then we have:

$$(1+\sqrt{2})[V] = (1+\sqrt{2})\begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1+\sqrt{2} \end{bmatrix}.$$

Indeed, the point is that multiplying a sum by a difference gives a difference of squares. In this situation, it gets rid of the square root in the place where we don't want it. And then we would work with this new vector instead of $[V]$. Similarly, if we wanted both square roots at the bottom, we could multiply $[U]$ by a suitably chosen sum.