Comments about Chapters 4 and 5 of the Math 5335 (Geometry I) text

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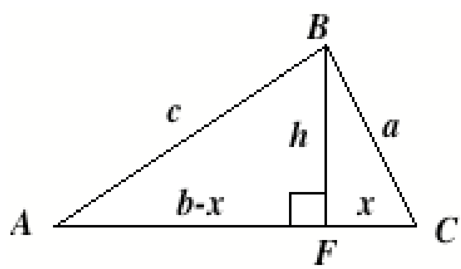
November 5, 2003; revised October 18, 2004 and October 2005

Contents**:**

1. ***Heron's formula (Theorem 9 in §4.5).*** Here is a proof that doesn't depend on the formulas from §4.4. The starting point is the observation that there are two different ways to use the Pythagorean Theorem to write a formula for h^2 in the following figure.



By definition, F is the foot of the altitude (perpendicular) drawn from B to $\overleftrightarrow{AC}$. As usual, a, b and c denote the lengths of the sides opposite the vertices $A, B,$ and C respectively. We define x to be the distance from C to F: positive in case F is in the ray $\overrightarrow{CA}$ and negative if not. (We would get a negative value if we had an obtuse angle at C.) Similarly, $b - x$ is the distance (with ± sign) from A to F.

With this setup, $h^2 = a^2 - x^2$, and also $h^2 = c^2 - (b - x)^2 = c^2 - b^2 + 2bx - x^2$. Setting these two expressions equal, we obtain:

$$a^2 - x^2 = c^2 - b^2 + 2bx - x^2,$$

so that:

$$2bx = a^2 + b^2 - c^2, \qquad \text{and therefore} \qquad x = (a^2 + b^2 - c^2)/2b\,.$$

Substituting this into the equation $h^2 = a^2 - x^2$, we obtain:

$$h^2 = a^2 - \left((a^2 + b^2 - c^2)^2/4b^2\right) = (2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)/4b^2\,.$$

This leads to a formula which is equivalent to formula (3.3) of the text:

$$\|\Delta ABC\|^2 = {}^1/_4\, b^2h^2 = (2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)/16\,.$$

And exactly as in the text, the numerator can be factored. One way to see that is to note that the numerator, namely $-(c^4 - 2(a^2 + b^2)c^2 + (a^4 - 2a^2b^2 + b^4))$, is of 4th degree in c with only even exponents and is thus a quadratic expression in c^2. If we set it equal to zero and regard it as a quadratic equation with c^2 as the unknown, then the quadratic formula produces the following roots:

$$c^2 = (a^2 + b^2) \pm \sqrt{4a^2b^2} = (a^2 \pm 2ab + b^2)$$

Therefore, we have the following factorization:

$$-(c^4 - 2(a^2 + b^2)c^2 + (a^4 - 2a^2b^2 + b^4)) = -(c^2 - (a + b)^2)(c^2 - (a - b)^2).$$

Since the last terms are differences of squares, this leads to:

$$-(c^4 - 2(a^2 + b^2)c^2 + (a^4 - 2a^2b^2 + b^4)) = -(c + (a + b))(c - (a + b))(c + (a - b))(c - (a - b)).$$

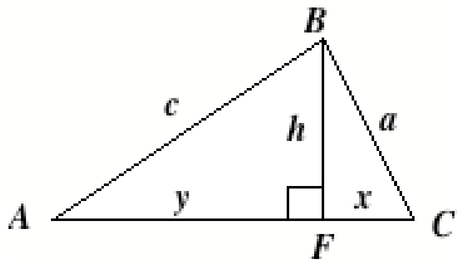
If the minus sign is absorbed into the second factor on the right, then we obtain a nearly final result:

$$\|\Delta ABC\|^2 = (a + b + c)(a + b - c)(a + c - b)(b + c - a))/16$$

To get a more traditional version of the formula, we set $s = (a + b + c)/2$ (sometimes called the ***semi-perimeter***) and then observe that $s - a = (a + b + c)/2$, and so forth, thus leading to:

$$\|\Delta ABC\|^2 = s(s - a)(s - b)(s - c), \qquad \text{or} \qquad \|\Delta ABC\| = \sqrt{s(s-a)(s-b)(s-c)}$$

2. ***§4.4: Another proof of Theorem 7.*** Here, we'll use a figure very similar to the one used in the proof of Heron's formula presented above. Indeed, the only change is that we've labeled the distance from



A to F as y instead of $b - x$, since we'll actually want to find its value.

As in the previous proof, we have two ways to calculate h namely $h^2 = a^2 - x^2$, and also $h^2 = c^2 - y^2$. Setting them equal to each other, we obtain the following equation:

$$a^2 - x^2 = c^2 - y^2.$$

And here is our other equation:

$$x + y = b.$$

Substituting $y = b - x$ into the first equation, we have $a^2 - x^2 = c^2 - b^2 + 2bx - x^2$. Thus:

$$2bx = a^2 + b^2 - c^2, \qquad \text{so that} \qquad x = (a^2 + b^2 - c^2)/2b\,.$$

By doing a similar calculation, or by setting $y = b - (a^2 + b^2 - c^2)/2b$, we obtain:

$$y = (b^2 + c^2 - a^2)/2b\,.$$

Now, what does this tell us about the barycentric coordinates of F? A preliminary guess might be that $F = (y/b, 0, x/b)^\Delta = ((b^2 + c^2 - a^2)/2b^2, 0, (a^2 + b^2 - c^2)/2b^2)^\Delta$, ***but this guess would be wrong!!*** Well, at a minimum, it would be the "opposite" of what's predicted in Theorem 6. To see why, and to determine which choice really is correct, consider the actual barycentric coordinates $F = (r,0,t)^\Delta$. In rectangular coordinates, this is $F = rA + tC$. So, to get the distance from A, we calculate:

$$F - A = (rA + tC) - A = (r - 1)A + tC = -tA + tC = t(C - A).$$

Thus, the signed(±) distance from A is tb. In other words, the distance from A to F is associated with the ***3rd barycentric coordinate***. And in a similar way the distance from B to F is associated with the ***1st barycentric coordinate***. Accordingly: $F = ((a^2 + b^2 - c^2)/2b^2, 0, (b^2 + c^2 - a^2)/2b^2)^\Delta$,

The results can summarized as in the following table. (The above calculation above gives the middle row.)

Vertex	opposite side	Foot of altitude
A	$\overline{BC}$	$(0, (a^2 + b^2 - c^2)/2a^2, (a^2 + c^2 - b^2)/2a^2)^\Delta$
B	$\overline{AC}$	$((a^2 + b^2 - c^2)/2b^2, 0, (b^2 + c^2 - a^2)/2b^2)^\Delta$
C	$\overline{AB}$	$((a^2 + c^2 - b^2)/2c^2, (b^2 + c^2 - a^2)/2c^2, 0)^\Delta$

3. ***§4.4: A proof of Theorem 8 (or the equivalent Corollary 10).*** Actually, this amounts to a solution of Problem 43, and is possibly "improper" material for that reason. But I'll proceed with it anyway. Let $E, F,$ and G be the feet of the altitudes passing through $A, B,$ and C respectively, and let $O = (r,s,t)^\Delta$ be the orthocenter. By the table, $E = (0, {}^{(a^2+b^2-c^2)}/_{2a^2}, {}^{(a^2+c^2-b^2)}/_{2a^2})^\Delta$. Since O is on the altitude through A, we have $(r,s,t) = u(1,0,0) + v(0, {}^{(a^2+b^2-c^2)}/_{2a^2}, {}^{(a^2+c^2-b^2)}/_{2a^2})$ in $\mathbb{R}^3$, where u and v are real numbers with $u + v = 1$. We can use this to get information about the ratio between s and t. More specifically, we look at the second and third components of the vectors in this equation to see that $({}^{(a^2+b^2-c^2)}/_{2a^2}, {}^{(a^2+c^2-b^2)}/_{2a^2})$ and (s,t) are linearly dependent in $\mathbb{R}^2$, thereby showing that the usual determinant is equal to zero:

$$\begin{vmatrix} {}^{a^2+b^2-c^2}\!\diagup_{2a^2} & {}^{a^2+c^2-b^2}\!\diagup_{2a^2} \\ s & t \end{vmatrix} = 0$$

It follows that $s(a^2 + c^2 - b^2) = t(a^2 + b^2 - c^2)$. Working in a similar way with the altitude through C, we show that $r(b^2 + c^2 - a^2) = s(b^2 + c^2 - a^2)$. These equations have very nice symmetry, but our "multiplier method" would work better with $r, s,$ and t divided (rather than multiplied) by various algebraic expressions. To accomplish this we divide ***both*** of our equations by the product $(b^2 + c^2 - a^2)(b^2 + c^2 - a^2)(a^2 + b^2 - c^2)$. This process yields the following equations:

$$\frac{r}{(a^2+c^2-b^2)(a^2+b^2-c^2)} = \frac{s}{(b^2+c^2-a^2)(a^2+b^2-c^2)} = \frac{t}{(b^2+c^2-a^2)(a^2+c^2-b^2)}$$

We can multiply out the denominators in the following manner:

$$\frac{r}{a^4-(b^2-c^2)^2} = \frac{s}{b^4-(a^2-c^2)^2} = \frac{t}{c^4-(a^2-b^2)^2}$$

so that:

$$\frac{r}{a^4-b^4-c^4+2b^2c^2} = \frac{s}{b^4-a^4-c^4+2a^2c^2} = \frac{t}{c^4-a^4-b^4+2a^2b^2}$$

Now, let λ be the common value of these ratios. Then we have $r = \lambda(a^4 - b^4 - c^4 + 2b^2c^2)$, $s = \lambda(b^4 - a^4 - c^4 + 2a^2c^2)$, and $t = \lambda(c^4 - a^4 - b^4 + 2a^2b^2)$. Substituting these equations into the basic identity $r + s + t = 1$, we obtain:

$$\lambda((a^4 - b^4 - c^4 + 2b^2c^2) + (b^4 - a^4 - c^4 + 2a^2c^2) + (c^4 - a^4 - b^4 + 2a^2b^2)) = 1,$$

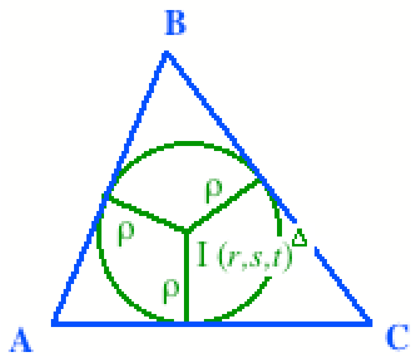
or simply:

$$\lambda(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4) = 1, \quad \text{so that} \quad \lambda = \frac{1}{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}$$

By one of the intermediate steps in the proof of Heron's formula, this becomes $\lambda = \dfrac{1}{16\|\Delta ABC\|^2}$

so that $O = \left(\dfrac{a^4-(b^2-c^2)^2}{16\|\Delta ABC\|^2}, \dfrac{b^4-(a^2-c^2)^2}{16\|\Delta ABC\|^2}, \dfrac{c^4-(a^2-b^2)^2}{16\|\Delta ABC\|^2}\right)^\Delta.$

4. ***§4.7: The barycentric coordinates of the incenter.*** We'll give a version of the proof of this formula (Theorem 17 on page 100) based on a slightly different viewpoint from what is presented in the text. Another significant feature is that ***the formula can be derived without using the existence of angle bisectors*** (Proposition 16 of §4.7). And in fact, the proof in the text also works from a similar frame of reference if you study it carefully.



The discussion that follows is based on the proof that was presented in class. Thus, we are looking for an incenter (whose existence we need to prove): a point in the interior of the triangle that is given in barycentric coordinates as $I = (r,s,t)^{\Delta}$. Thus, we need to find formulas for the barycentric coordinates, under the assumption that r, s, and t all are positive. We denote the radius of the inscribed circle as ρ (rather than r) to avoid confusion with the first barycentric coordinate. Accordingly, ρ is the distance of I from each side of the triangle.

By Theorem 32 of Chapter 3, the distance of $I = (r,s,t)^{\Delta}$ from $\overleftrightarrow{AC}$ is $h_B s$, where h_B is the distance from B to $\overleftrightarrow{AC}$, ***i.e.,*** the length of the altitude from B to $\overleftrightarrow{AC}$. Thus, we have the equation $\rho = h_B s$. {Literally applied, the theorem requires multiplying by $|s|$ rather than s, but $s > 0$ because we are looking for a point in the interior of the triangle.} In a similar way, we obtain the equations $\rho = h_A r$ and $\rho = h_C t$. If we eliminate ρ from these three equations, we are left with the following two equations:

$$h_A r = h_B s \quad \text{and} \quad h_B s = h_C t.$$

To get an equation that relates the altitudes to other known quantities, we observe (for instance) that we have the relation $\|\Delta ABC\| = h_B b/2$, which gives $h_B = 2\|\Delta ABC\|/b$. Using this, along with similar equations that involve the other altitudes, we can transform our equations into:

$$2r\|\Delta ABC\|/a = 2s\|\Delta ABC\|/b \quad \text{and} \quad 2s\|\Delta ABC\|/b = 2t\|\Delta ABC\|/c,$$

or after canceling common factors:

$$r/a = s/b \quad \text{and} \quad s/b = t/c.$$

Along with these two equations, we have the standard equation $r + s + t = 1$. One way to solve the resulting system involves setting λ equal to the common value of the ratios r/a, s/b and t/c. Thus, $\lambda = r/a = s/b = t/c$ so that $r = a\lambda$, $s = b\lambda$, and $t = c\lambda$. Substituting these values into the equation $r + s + t = 1$, we find that $(a + b + c)\lambda = 1$, so that $\lambda = \dfrac{1}{a+b+c}$. If we substitute this into our previous equations, then we obtain the following formula:

$$I = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)^{\Delta}.$$

This certainly identifies a point in the interior of ΔABC which is equidistant from the 3 sides of the triangle.

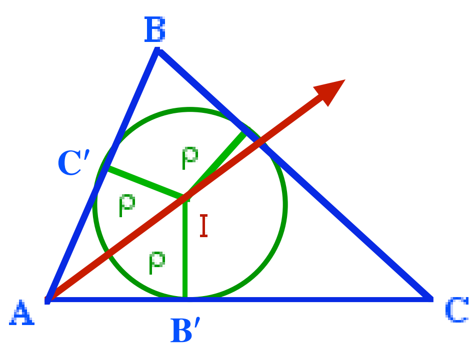
Incidentally, we also can combine this with the equations $\rho = h_B s$ and $h_B = 2\|\Delta ABC\|/b$ to derive the formula $\rho = \dfrac{2\|\Delta ABC\|}{a+b+c}$, also given in the text as part of Theorem 16.

5. ***§4.7: The existence of angle bisectors (An alternative approach).*** This result corresponds to Proposition 16 on page 98 of the text. By definition, an ***angle bisector*** of $\angle BAC$ is a ray $\overrightarrow{AQ}$ in the interior of $\angle BAC$ such that $\angle BAQ \cong \angle CAQ$. {And because Q is in the interior of the angle, it follows that $|\angle BAQ|$ and $|\angle CAQ|$ both are equal to $|\angle BAC|/2$.} Proposition 16 asserts that an angle bisector exists and is unique. Proving that result (or at least mentioning it) before discussing Theorem 17 certainly is the more traditional approach. On the other hand, showing that a ray with the required angle congruence properties lies in the ***interior*** of the angle is somewhat intricate. This (somewhat subtle) question seems generally to be ignored in high school geometry courses, probably to avoid confusing students who are learning geometry for the first time.

On the other hand, having already proven our result about the incenter, we actually have a somewhat less intricate means available to prove the existence of the angle bisector:

Proposition. If I is the incenter of ABC, then $\overrightarrow{AI}$ is the angle bisector of $\angle BAC$

PROOF. To get started, let's recall that we've shown that I is in the interior of $\angle BAC$, so it follows that the entire ray $\overrightarrow{AI}$ is contained in the interior of this angle. Next, let B' be the point of $\overleftrightarrow{AC}$ closest to I, and let C' be the point of $\overleftrightarrow{AB}$ closest to I.



Then the triangles $\Delta AB'I$ and $\Delta AC'I$ have right angles at B' and C' respectively. Hence, we can apply the Pythagorean theorem to show that $|\overline{AB'}|^2 = |\overline{AI}|^2 - |\overline{B'I}|^2 = |\overline{AI}|^2 - \rho^2$ and similarly that $|\overline{AC'}|^2 = |\overline{AI}|^2 - \rho^2$. In particular, we have $|\overline{AC'}| = |\overline{AB'}|$. Therefore, we can apply the SSS criterion to conclude that $\Delta AB'I \cong \Delta AC'I$. It follows that there is an isometry that maps A to itself, I to itself, and B' to C. We conclude that $\angle BAI \cong \angle CAI$, so that we have an angle bisector, as claimed.

Remark. The isometry constructed in the previous proof maps the ray $\overrightarrow{AI}$ to itself and maps the ray $\overrightarrow{AB'}$ to the ray $\overrightarrow{AC'}$. We can use this fact as a starting point of proving that every point of $\overrightarrow{AI}$ is equidistant from the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

6. ***Chapter 5: The inner product and the cosine.*** In Chapter 2, we defined angular measure of an angle, whose sides are rays with direction indicators U and V, to be equal to the integral $\displaystyle\int_{\langle U,V\rangle}^{1} \frac{ds}{\sqrt{1-s^2}}$, We also observed that this expression defines the integral as a strictly decreasing

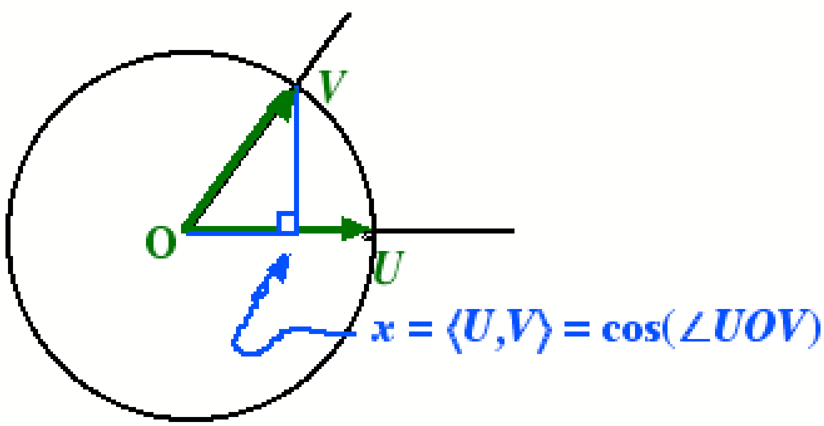
function of its lower endpoint, and we decided to call this function the ***arccosine***. Thus, the arccosine turns out to be a strictly decreasing function that maps the closed interval [-1,1] to the closed interval [0,π]. Since a strictly increasing function or a strictly decreasing function is a bijective mapping from its domain to its range, it has an inverse function. The inverse function of the arccosine function is the ***cosine***. Thus, the cosine is a strictly decreasing function that maps the interval [0,π] to the interval [-1,1]. So, if $\theta = \int_{\langle U,V\rangle}^{1} \frac{ds}{\sqrt{1-s^2}} = \arccos(\langle U,V\rangle)$ is the measure of our angle, then we have $\langle U,V\rangle = \cos(\theta)$ in the case where U and V are unit vectors.

More generally, if U and V are direction indicators of the sides of an angle (but not necessarily unit vectors), and if θ is the angular measure, then we have the following important identity:

$$\langle U,V\rangle = \|U\|\,\|V\|\cos\theta,$$

which may be familiar from vector calculus courses. To check it in our situation, we observe that $U = aU_0$ and $V = bV_0$, where U_0 and V_0 are unit vectors, and a and b are ***positive*** real numbers. (We want U_0 and V_0 to point in the same direction as U and V respectively.) So, $\|U\| = a$, and $\|V\| = b$, while $\langle U,V\rangle = ab\langle U_0,V_0\rangle$. Therefore, the identity $\langle U,V\rangle = \|U\|\,\|V\|\cos\theta$ follows from the previously known formula $\langle U_0,V_0\rangle = \cos(\theta)$.

7. ***Chapter 5: The cosine function, the inner product, and right angle trigonometry.*** Historically, the most basic definition of the cosine of an angle was the quotient of adjacent side over hypotenuse in a right triangle. In our formulation, this is fairly immediate in the case where the unit vector (1,0) is one of the sides of an angle.



To check this algebraically, we set $U = (1,0)$ and $V = (v_1,v_2)$ and then calculate: $\langle U,V\rangle = \langle(1,0),(v_1,v_2)\rangle = v_1$. Thus, $(v_1,0) = v_1U = \langle U,V\rangle U$ turns out to be at the base of the perpendicular from the point V to the line $\overleftrightarrow{OU}$. A negative value of $\langle U,V\rangle$ is interpreted as meaning that the base of this perpendicular lies on the ray opposite to $\overrightarrow{OU}$.

8. ***5.4: About the law of cosines.*** As noted in the text, the proof of the "first version" of the Law of Cosines uses Lemma 42 of Chapter 1. Since that lemma isn't proved in the text, we'll first state and prove a variant of that auxiliary result.

Lemma. Let A, B, and C be points in $\mathbf{R}^n$. Then:

$$\|A-B\|^2 = \|A-C\|^2 + \|B-C\|^2 - 2\langle(A-C),(B-C)\rangle$$

Proof: We recall that $\|A-B\|^2 = \langle(A-B),(A-B)\rangle$ and then insert the "missing term", namely C, to make A-B appear as a difference of differences. More plainly, the idea is to write:

$$A\text{-}B = (A\text{-}C) - (B\text{-}C).$$

Using this, we calculate the inner product:

$$\langle (A\text{-}B),(A\text{-}B)\rangle = \langle (A\text{-}C) - (B\text{-}C), (A\text{-}C) - (B\text{-}C)\rangle$$
$$= \langle (A\text{-}C), (A\text{-}C)\rangle - \langle (A\text{-}C),(B\text{-}C)\rangle - \langle (B\text{-}C), (A\text{-}C)\rangle + \langle (B\text{-}C),(B\text{-}C)\rangle.$$

{Formally, we used the fact that the inner product is linear in each of the variables. More informally, we can view it as similar to expanding the binomial expression $(X\text{-}Y)^2$.} Next, we can use the symmetry of the inner product: $\langle U,V\rangle = \langle V,U\rangle$ to obtain:

$$\langle (A\text{-}B),(A\text{-}B)\rangle = \langle (A\text{-}C), (A\text{-}C)\rangle - 2\langle (A\text{-}C),(B\text{-}C)\rangle + \langle (B\text{-}C),(B\text{-}C)\rangle.$$

Finally, we replace each inner product $\langle U,U\rangle$ with the square of the corresponding norm to obtain:

$$\|A\text{-}B\|^2 = \|A\text{-}C\|^2 + \|B\text{-}C\|^2 - 2\langle (A\text{-}C),(B\text{-}C)\rangle,$$

thus proving the lemma.

To apply this when we're thinking of A, B, and C as the vertices of a triangle in $\mathbf{R}^2$, we use the letters a, b, and c to denote the lengths of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively. Thus:

□ $$c = |\overline{AB}| = \|A\text{-}B\|,$$

and so forth. If we make these substitutions we obtain the identity:

$$c^2 = a^2 + b^2 - 2\langle (A\text{-}C),(B\text{-}C)\rangle.$$

As a final step, we apply the identity $\langle U,V\rangle = \|U\|\,\|V\| \cos\theta$ from the previous section, with $U = A\text{-}C$, $V = B\text{-}C$, and $\theta = |\angle ACB| = |\angle C|$ to obtain the following identity:

$$c^2 = a^2 + b^2 - 2\|A\text{-}C\|\cdot\|B\text{-}C\| \cos(C),$$

from which we deduce the law of cosines.

Law of Cosines (1st version). Given ΔABC, let $a = |\overline{BC}|$, $b = |\overline{AC}|$, and $c = |\overline{AB}|$. Then:

$$c^2 = a^2 + b^2 - 2ab\cos(C),$$

Just to check that our answer makes sense at least in a special case, note that if $|\angle C| = \pi/2$, then $\cos(C) = 0$, and we recover the usual Pythagorean identity $c^2 = a^2 + b^2$.

Finally, we can transform our main identity algebraically to obtain the other version of the law of cosines.

Law of Cosines (2nd version). Given ΔABC, let $a = |\overline{BC}|$, $b = |\overline{AC}|$, and $c = |\overline{AB}|$. Then:

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab},$$

In particular, if $c^2 < a^2 + b^2$, this gives a positive value of $\cos(C)$, corresponding to an acute angle at C. On the other hand, if $c^2 > a^2 + b^2$, the formula gives a negative value of $\cos(C)$, corresponding to an obtuse (***i.e.,*** non-acute) angle at C.

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