Comments about Chapter 2 of the Math 5335 (Geometry I) text

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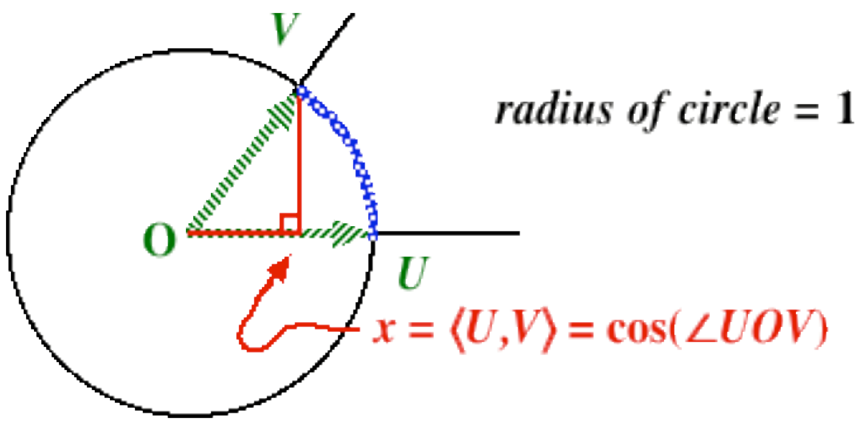
Contents**:**

- *§2.2: angles with a given angular measure.*
- *Preview of Chapter 3: another viewpoint about angular measure.*
- *§2.4: the outward angles of a triangle*

1. ***§2.2: angles with a given angular measure.*** (This applies to the proof of Proposition 5, and also could be useful in problems that involve finding an isometry that maps one angle to another one.) The relevant statement is "…there are two rays emanating from (0,0) each of which together with the ray $\overrightarrow{(0,0)(1,0)}$ forms an angle of specified measure".

   In many situations, such as when we're trying to duplicate a given angle, the angular measure is given in the form $\int_{\langle U,V\rangle}^{1} \frac{ds}{\sqrt{1-s^2}}$, where U and V are the unit direction vectors of the rays that form the two sides of the angle. From one of the figures in the §1.4 and §1.5 text supplement, this is how it looks when one of the two unit vectors points in the direction of the positive x-axis:

   With slightly less elaborate notation, we can get an arc with length $= \int_{a}^{1} \frac{ds}{\sqrt{1-s^2}}$ (and thus an angle with this angular measure) by choosing the left-hand endpoint of the arc to have x-coordinate $= a$. In other words, we choose the x-coordinate to be equal to the lower limit of the integration. Thus, we have $U = (1,0)$ in any case. Depending on whether we want our angle to be in the upper half plane or in the lower half plane, we take $V = (b, \sqrt{1-b^2})$ or $V = (b, -\sqrt{1-b^2})$ respectively. (Note that V is to be a vector of norm = 1 or a point on the unit circle, according to one's preferences, and hence the 2nd coordinate must be $\pm\sqrt{1-b^2}$.)

2. ***Preview of Chapter 3: another viewpoint about angular measure.*** In the above, if the left-hand end of the arc is at $x = a,$ then the arc length, or angular measure, is $\int_a^1 \frac{ds}{\sqrt{1-s^2}}$. Thus, the equation $\theta = \int_a^1 \frac{ds}{\sqrt{1-s^2}}$ defines θ as a function of a. Because the integrand $\frac{1}{\sqrt{1-s^2}}$ is always positive and we're regarding the integral as a function of a variable ***lower*** limit, θ is a ***strictly decreasing*** function of a. In §3.10, we'll define this as being $\theta = \arccos(a)$. Thus, we can write $\arccos(a) = \int_a^1 \frac{ds}{\sqrt{1-s^2}}$.

Since our arccosine function is a strictly decreasing function from the interval [-1,1] to the interval [0,π], it is bijective and therefore has an inverse function. (To see that the interval is what we claim, note that $\arccos(1) = 0$, while $\arccos(-1) = \int_{-1}^1 \frac{ds}{\sqrt{1-s^2}} = \pi$ -- the arc length of a semicircle.) So, we define the cosine to be the inverse function -- at least on the interval [0,π]. Thus, $\cos(0) = 1$, while $\cos(\pi) = -1$. We can get a value on the interior of the interval, as follows:

$$\arccos(0) = \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{2}$$ -- the arc length of a quarter circle.

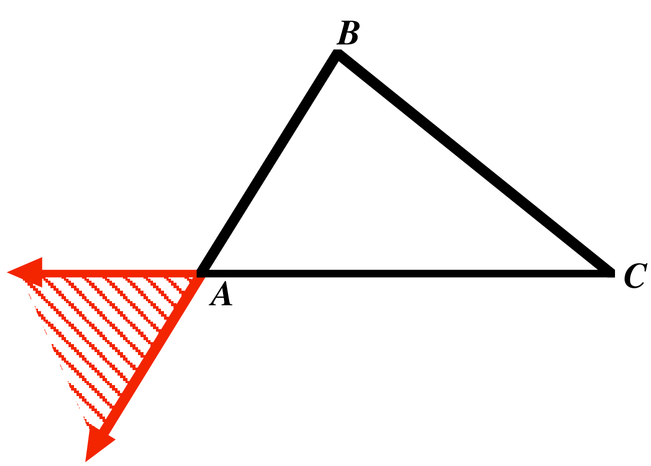
Accordingly, $\cos\left(\frac{\pi}{2}\right) = 0$ -- once again the result that we would have expected.

Notes:

(1) The equation $\theta = \int_a^1 \frac{ds}{\sqrt{1-s^2}}$ is a ***definition*** rather than a theorem since we're trying to develop geometry without basing our proofs on previous knowledge of trigonometry.

(2) The indefinite integral $\int \frac{ds}{\sqrt{1-s^2}}$ usually is associated with the arcsine function, and a definite integral also would be treated similarly ***if it were*** being considered as a function of its upper limit. But since we're considering it as a function of its lower limit, it's appropriate to consider it as the arccosine instead.

3. ***§2.4: the outward angles of a triangle.*** By definition, the ***outward angle*** at A of ΔABC consists of the rays opposite to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The interior of this outward angle is called the ***outward angular region.*** This is illustrated in the following figure, where the two opposite rays are shown in red, and a finite portion of the exterior angular region is lightly shaded in red.



Go back:

To the class homepage

To the homework page

To the course description