Math 5335 Section 2 Fall 2005
Solutions to December 8 homework problems

PROBLEM 9.17 To find any intersection points, we have to solve the following system of equations:

$$x^2 + y^2 = 6,$$
$$(x - 1)^2 + y^2 = 1.$$

We expand (and slightly simplify) the second equation to $x^2 - 2x + y^2 = 0$. Substituting into this from the first equation, we have $-2x + 6 = 0$, or $x = 3$. If we substitute this into the first equation, we obtain $y^2 = -3$. Since this has no real solutions, we conclude that there are ***no intersection points***.
Note: If we plot the two circles, we see that the smaller one [with center (1,0) and radius = 1] is completely contained inside the larger one [with center (0,0) and radius = $\sqrt{6}$]. Thus the circles do not intersect, which certainly is consistent with the solution presented here. But a complete solution definitely involves doing the algebra, or somehow ***proving*** that the circles don't intersect.

PROBLEM 9.20. The figure and the equations of the lines are exactly as they were given in the back of the text. To describe the Poincaré segments, we also need inequalities to specify the range of x-coordinates. Thus:

Segment	***Equation of Poincaré line***	***x-range of segment***
$\overline{(-2,4)(2,4)}$	$x^2 + y^2 = 20,\ y > 0$	$-2 \le x \le 2$
$\overline{(-2,4)(0,4)}$	$(x+1)^2 + y^2 = 17,\ y > 0$	$-2 \le x \le 0$
$\overline{(0,4)(2,4)}$	$(x-1)^2 + y^2 = 17,\ y > 0$	$0 \le x \le 2$

PROBLEM 10.5. The circle that gives the Poincaré line has center (- 4, 0) and radius 2. Therefore it intersects the x-axis at the points (- 6, 0) and (- 2, 0). These two points are therefore the direction indicators of the Poincaré line $(x + 4)^2 + y^2 = 4,\ \ y > 0$. The one on the left belongs to the ray given by the inequality $x \le -3$, and the one on the right belongs to the ray given by the inequality $x \ge -3$. Thus:

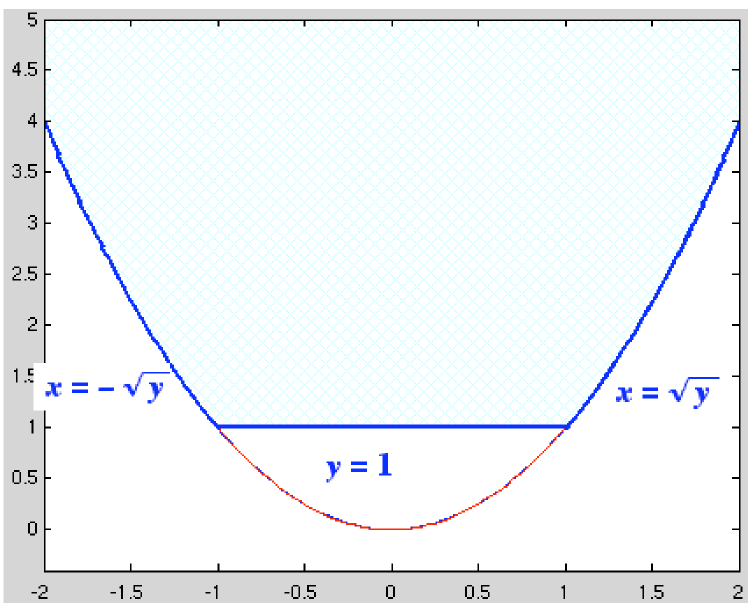
Ray	***Direction indicator***
$(x + 4)^2 + y^2 = 4,\ \ x > -3$	(–6, 0)
$(x + 4)^2 + y^2 = 4,\ \ x > -3$	(–2, 0)

Note: Literally correct descriptions of the Poincaré rays could be understood to include the inequality $y > 0$ as well as the inequality that specifies the range of x-values. But that is often omitted because it is understood that we're working in the upper half plane.

PROBLEM 10.12. The Poincaré line $(x-2)^2 + y^2 = 1, y > 0$ has direction indicators (1, 0) and (3,0), while the Poincaré line $(x+2)^2 + y^2 = 1, y > 0$ has direction indicators (-1, 0) and (-3, 0). Any line which is asymptotically parallel to both of these must have ***one direction indicator in common with each***. Here are the possibilities:

Direction indicators	**(ω, 0)**	ρ	***Poincaré line***
(1, 0) and (–1, 0)	(0, 0)	1	$x^2 + y^2 = 1, \quad y > 0$
(3, 0) and (–3, 0)	(0, 0)	$\sqrt{3}$	$x^2 + y^2 = 3, \quad y > 0$
(3, 0) and (–1, 0)	(1, 0)	2	$(x-1)^2 + y^2 = 4, \quad y > 0$
(1, 0) and (–3, 0)	(–1, 0)	2	$(x+1)^2 + y^2 = 4, \quad y > 0$

PROBLEM 10.28. The expression $\max(x^2, 1)$ refers to the larger of the two numbers x^2 and 1. Therefore, the inequality $y \geq \max(x^2, 1)$ describes the set of all points that are above the parabola $y = x^2$ and also above the line $y = 1$. It is shown in the following figure:

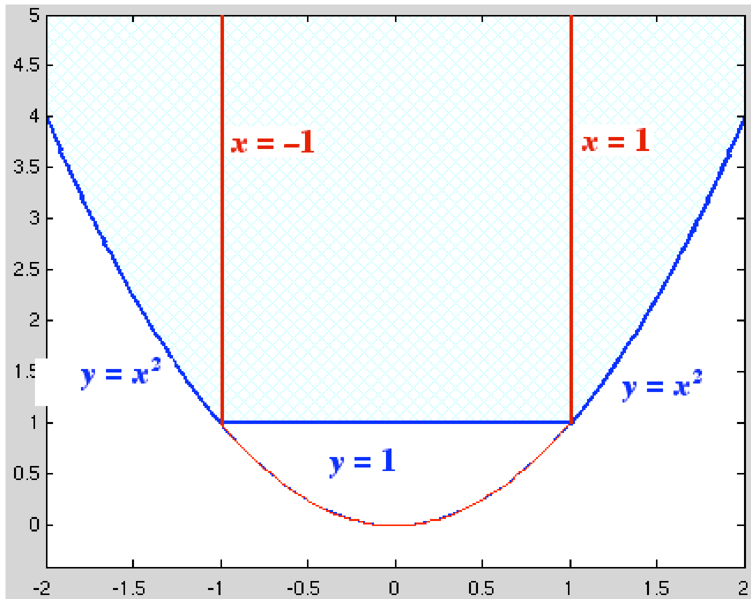


Generally, the hyperbolic area of a region Ω is given as the integral $\iint_{\Omega} \frac{dxdy}{y^2}$.

If we want to do the integral in a single piece, then we use the "backwards(?)" order of integration, ***i.e.,*** we integrate first with respect to *x,* treating the right side of the parabola as being given by the equation $x = \sqrt{y}$ and the left side of the parabola as being given by the equation $x = -\sqrt{y}$. In this way, we get the following integral:

$$\int_1^{\infty}\left(\int_{-\sqrt{y}}^{\sqrt{y}} \frac{dx}{y^2}\right)dy = \int_1^{\infty} \frac{x}{y^2}\bigg|_{x=-\sqrt{y}}^{x=\sqrt{y}} dy = \int_1^{\infty} 2y^{-3/2}dy = -4y^{-1/2}\Big|_{y=1}^{y=\infty} = 4.$$

Alternatively, if we wish to do the "more traditional(?)" order of integration, where we integrate first with respect to y (and therefore last with respect to x), then we have to divide the region into three pieces, as shown in the following figure:



By symmetry, the region on the right has the same area as the region on the left. Therefore, the area is given by the following sum of integrals:

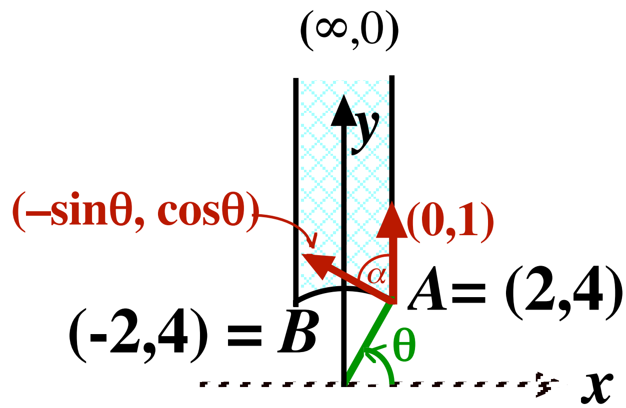
$$\int_{-1}^{1}\left(\int_{1}^{\infty}\frac{dx}{y^2}\right)dx+2\int_{1}^{\infty}\left(\int_{x^2}^{\infty}\frac{dy}{y^2}\right)dx = \int_{-1}^{1}\frac{-1}{y}\bigg|_{y=1}^{y=\infty}dx+2\int_{1}^{\infty}\frac{-1}{y}\bigg|_{y=x^2}^{y=\infty}dx$$

$$=\int_{-1}^{1}dx+2\int_{1}^{\infty}\frac{dx}{x^2} = 2+2\frac{-1}{x}\bigg|_{x=1}^{x=\infty} = 2+2=4.$$

{Please see the next page for Problem 10.31.}

PROBLEM 10.31. We calculate the hyperbolic area as $\pi - \alpha - \beta$, where α and β are the angular measures at the non-asymptotic vertices. By symmetry (see the figures below), we have $\alpha = \beta$, so that the hyperbolic area is $\pi - 2\alpha$.

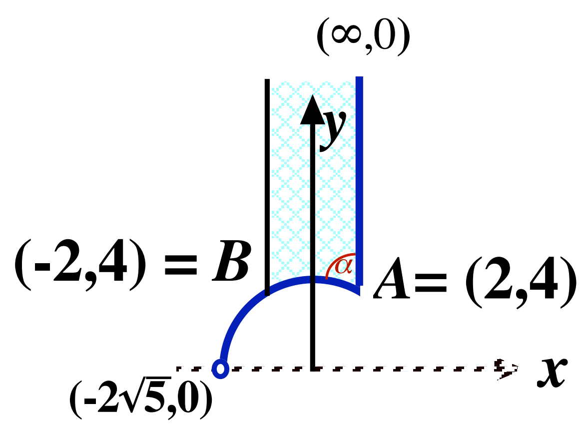
Method 1. We find $\cos\alpha$ by calculating the inner product of the two tangent vectors at A. Clearly, (1,0) is the unit tangent vector of the horizontal ray. The radius of the circular arc is $= 2\sqrt{5}$, so that it is given parametrically as $X(t) = 2\sqrt{5}(\cos t, \sin t)$. Hence, we find a tangent vector at $A = X(\theta)$ by calculating $X'(\theta) = 2\sqrt{5}(-\sin\theta, \cos\theta)$. Therefore, the ***unit*** tangent vector is $(-\sin\theta, \cos\theta)$, as indicated in the figure. And then we calculate:



$$\cos\alpha = \langle(-\sin\theta, \cos\theta), (0,1)\rangle = \cos\theta.$$

Referring to the figure and noting that the circle has radius $= 2\sqrt{5}$, we see that $\cos\theta = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$. Therefore, $\cos\alpha = \frac{1}{\sqrt{5}}$, so that $\alpha = \arccos\frac{1}{\sqrt{5}}$. Hence the hyperbolic area is $\pi - 2\arccos\frac{1}{\sqrt{5}}$. Alternatively, we can write it as $\pi - 2\arcsin\frac{2}{\sqrt{5}}$. When finding the numerical value, remember to set your calculator for ***radians***, rather than degrees. Thus, the ***numerical value*** is 0.927295.

Method 2. Again, we have symmetry, so that the hyperbolic area is $\pi - 2\alpha$. The Poincaré rays that form the sides of the angle are shown in dark blue in the figure. We use the formula from Theorem 7 in §10.3, with $(g,h) = (2,4)$, the direction indicators of the two rays being $(-2\sqrt{5}, 0)$ and $(\infty, 0)$. Therefore, we apply version (10.4) of the formula, as follows:



$$\alpha = \arccos\left(\frac{(-2\sqrt{5}-2)^2 - 4^2}{(-2\sqrt{5}-2)^2 + 4^2}\right)$$

$$= \arccos\left(\frac{(\sqrt{5}+1)^2 - 2^2}{(\sqrt{5}+1)^2 + 2^2}\right)$$

We calculate $(\sqrt{5}+1)^2 = 5+2\sqrt{5}+1 = 6+2\sqrt{5}$. Therefore, the formula works out as follows:

$$\alpha = \arccos\left(\frac{2+2\sqrt{5}}{10+2\sqrt{5}}\right) = \arccos\left(\frac{1+\sqrt{5}}{5+\sqrt{5}}\right).$$

While this ***looks different*** from the previous answer, we can show (optionally) that it's really the same, as follows:

$$\frac{1+\sqrt{5}}{5+\sqrt{5}} = \frac{1+\sqrt{5}}{5+\sqrt{5}}\cdot\frac{5-\sqrt{5}}{5-\sqrt{5}} = \frac{5-\sqrt{5}+5\sqrt{5}-5}{25-5} = \frac{4\sqrt{5}}{20} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}.$$

We can describe the beginning of this process by saying that we multiplied numerator and denominator by the ***"conjugate"*** of the denominator. While it resembles the complex conjugate, it's a slightly different entity, namely the conjugate in the ***quadratic number field*** $\mathbf{Q}(\sqrt{5})$. {This is the field of rational numbers with $\sqrt{5}$ adjoined; elements of $\mathbf{Q}(\sqrt{5})$ are of the form $a+b\sqrt{5}$, where a and b are rational numbers. Obviously, sums and differences exist in this algebraic system. A bit of calculation is needed in order to check that products and quotients also exist ***inside*** the system. Our method of simplification above does, however, exhibit the method needed for proving that quotients exist.}

Back to the class homepage.