Math 5336
Spring 2004

Projective duality, intersections, and joins

Given a statement about points, lines, and incidence in $\mathbb{P}^2$, we obtain the dual statement by replacing "point" with "line" and "line" with point. Thus, suppose that we're given the following statement:

"three given distinct points are collinear",

We can put it more explicitly as follows:

"there is a line that it is incident with three given distinct points".

Therefore, the dual has to be the statement that:

"three given distinct lines are concurrent",

Indeed, the more explicit version of this is the following statement:

"there is a point that is incident with three given distinct lines".

And so, in the more explicit statements we see the exact substitution of "line" for "point" and "point" for line. In the more "idiomatic" versions we see that "concurrent" has been substituted for "collinear" – and clearly, when appropriate we would do the opposite substitution too.

Now, here's a more substantial question of the same nature: ¿when dualizing a statement, what substitution do we make for "intersection of distinct lines" or "join of distinct points"? Well, let's try a specific instance, namely the statement about concurrence that is one of the equivalent conditions mentioned in Desargues' Theorem in the plane. ***Please recall*** that a triangle in the projective plane consists of ***six*** parts, namely three non-collinear points and the three lines that join pairs of these points. Thus, for instance, if A, B and C are the vertices, we can denote the sides as $a = \overleftrightarrow{BC}$, $b = \overleftrightarrow{AC}$, and $c = \overleftrightarrow{AB}$

Thus, given two triangles $\triangle ABC$ and $\triangle A'B'C'$ in $\mathbb{P}^2$, we consider the statement:

(*) The lines joining corresponding vertices, namely $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$, are distinct and concurrent.

I claim that the following is its dual:

(**) The points formed by intersecting corresponding sides, namely $a \cap a'$, $b \cap b'$, and $c \cap c'$ are distinctand collinear.

Now, why is this correct? Well, we can translate (*) into the following more explicit version:

($*_E$) If $\mathcal{K}$ is the line incident with A and A', $\mathcal{L}$ is the line incident with B and B', and $\mathcal{M}$ is the line incident with C and C', then there is a point that is incident with all three of the lines $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{M}$.

And similarly, we can translate (**) into the following more explicit version:

> ($**_E$) If P is the point incident with a and a', Q is the point incident with b and b', and R is the point incident with c and c', then there is a point that is incident with all three of the points P, Q, and R.

So, finally to see that we have dualized correctly, we just need to observe that we have substituted "point" for "line" and "line" for "point" in the explicit versions.