(1) Consider the function \( f(x, y) = 4y\sqrt{x} \), find the maximum rate of change and the direction it occurs in.

The maximum rate of change is \( |\nabla f| \) and it occurs in the direction of \( \nabla f \) since \( D_u(F) = \nabla f \cdot u = |f| \cos \theta \).

We compute: \( \nabla f = \langle 2y\sqrt{x}, 4\sqrt{x} \rangle \). Evaluating at \((4, 1)\) gives \( \nabla f = \langle 1, 8 \rangle \). So, \( |\nabla f| = |\langle 1, 8 \rangle| = \sqrt{1+64} = \sqrt{65} \).

Thus the maximum rate of change is \( \sqrt{65} \) and occurs in direction \( \langle 1, 8 \rangle \).

(2) Find three positive numbers which sum to 100 and whose product is maximum (Hint: Rewrite \( x + y + z = 100 \) to eliminate \( z \)).

We need that \( x + y + z = 100 \) and that the product \( xyz \) is maximum. We first rewrite \( z = 100 - x - y \).

Then, our function to maximize is \( f(x, y) = xy(100 - x - y) \).

Then, \( \nabla f = \langle 100y - 2xy - y^2, 100x - x^2 - 2xy \rangle = 0 \). Setting \( f_x = 0 \) gives that \( y(100 - 2x - y) = 0 \). So \( y = 0 \) or \( y = 100 - 2x \). Substituting \( y = 0 \) into \( f_y \) gives that \( x = 0 \) or \( x = 100 \). Substituting \( y = 100 - 2x \) in gives \( 3x^2 - 100x = 0 \) i.e. \( x = 0 \) or \( x = 100/3 \). Thus, the possible critical points are \((0, 0), (100, 0), (0, 100), (\frac{100}{3}, \frac{100}{3})\).

We calculate that \( f_{xx} = -2y, f_{yy} = -2x \) and \( f_{xy} = 100 - 2x - 2y \).

So \( D(0,0) = D(100,0) = D(0,100) = -10000 \) and \( D(\frac{100}{3}, \frac{100}{3}) = \frac{10000}{3} \). Since \( f_{xx}(\frac{100}{3}, \frac{100}{3}) = -\frac{200}{3} \), we get that \((\frac{100}{3}, \frac{100}{3})\) is a maximum, which corresponds to having numbers \( x = y = z = \frac{100}{3} \).