Math 3283W: solutions to skills problems due 11/6

(13.7(abc)) All three statements can be proven false using counterexamples that you studied on last week’s skills homework.

(a) Let \( S = \{ \frac{1}{n} | n \in \mathbb{N} \} \). Every point of \( S \) is an isolated point, since given \( \frac{1}{n} \in S \), there exists a neighborhood \( N(\frac{1}{n}; \varepsilon) \) of \( \frac{1}{n} \) that contains no point of \( S \) different from \( \frac{1}{n} \) itself. (We could, for example, take \( \varepsilon = \frac{1}{n} - \frac{1}{n+1} \).) Thus the set \( P \) of isolated points of \( S \) is just \( S \). But \( P = S \) is not closed, since 0 is a boundary point of \( S \) which nevertheless does not belong to \( S \) (every neighborhood of 0 contains not only 0 but, by the Archimedean property, some \( \frac{1}{n} \), thus intersects both \( S \) and its complement); if \( S \) were closed, it would contain all of its boundary points.

(b) Let \( S = \mathbb{N} \). Then (as you saw on the last skills homework) \( \text{int}(S) = \emptyset \). But \( \emptyset \) is closed, so its closure is itself; that is, \( \text{cl}(\text{int}(S)) = \text{cl}(\emptyset) = \emptyset \), which is certainly not the same thing as \( S \).

(c) Let \( S = \mathbb{R} \setminus \mathbb{N} \). The closure of \( S \) is all of \( \mathbb{R} \), since every natural number \( n \) is a boundary point of \( \mathbb{R} \setminus \mathbb{N} \) and thus belongs to its closure. But \( \mathbb{R} \) is open, so its interior is itself; that is, \( \text{int}(\text{cl}(S)) = \text{int}(\mathbb{R}) = \mathbb{R} \), which is certainly not the same thing as \( S \).

(13.10) Suppose that \( x \) is an isolated point of \( S \), and let \( N = (x - \varepsilon, x + \varepsilon) \) (for some \( \varepsilon > 0 \)) be an arbitrary neighborhood of \( x \). To show \( x \) is a boundary point of \( S \), we must show two things: first, that \( N \) contains a point of \( S \), and second, that \( N \) contains a point of \( \mathbb{R} \setminus S \). The first is easy: part of the definition of “\( x \) is an isolated point of \( S \)” is that \( x \in S \), so the neighborhood \( N \) contains the point \( x \) of \( S \). The second is trickier. Since \( x \) is an isolated point of \( S \), it is not an accumulation point of \( S \), so (spelling out the negation of the definition of “accumulation point”) we see that there exists some deleted neighborhood \( N^\ast(x; \eta) = (x - \eta, x + \eta) \setminus \{ x \} \) of \( x \) (for some \( \eta > 0 \)) that contains no point of \( S \). We need to use the fact that \( (x - \eta, x + \eta) \) contains no point of \( S \) other than \( x \) itself to prove that our original neighborhood \( N = (x - \varepsilon, x + \varepsilon) \) contains at least one point of \( \mathbb{R} \setminus S \). The idea is that, regardless of whether \( \varepsilon \) or \( \eta \) is larger (we have no way of knowing which one), these are two open intervals with positive radius centered at the same point \( x \), so they overlap; and any of the points in the (nonempty) intersection \( N^\ast(x; \eta) \cap N \) is a point in the complement of \( S \). If you want an explicit formula for such a “bad” point, let \( \delta = \min(\eta, \varepsilon) \); then \( x + \frac{\delta}{2} \) belongs both to \( N \) and to \( S \).
(14.4) First, a preliminary result:

**Lemma.** If \( A \subset B \) are sets of real numbers and \( B \) is bounded, then \( A \) is bounded.

**Proof** of lemma. To say \( B \) is bounded is to say there exist an upper bound \( m \) and a lower bound \( \ell \) for \( B \). Given any \( x \in A \), we know \( x \in B \) since \( A \) is contained in \( B \). But then \( x \leq m \) since \( m \) is an upper bound for \( B \), and \( \ell \leq x \) since \( \ell \) is a lower bound for \( B \). That is, \( \ell \leq x \leq m \) for every \( x \in A \), so \( \ell \) (resp. \( m \)) is a lower (resp. upper) bound for \( A \) as well; that is, \( A \) is bounded, having both a lower bound and an upper bound.

Now we prove the claim of 14.4: the intersection of any collection of compact sets is compact. Let \( \{ C_i \}_{i \in I} \) be a family of compact sets. By the Heine-Borel theorem, we know that for subsets of \( \mathbb{R} \), “compact” is equivalent to “closed and bounded”. So each \( C_i \) is closed and bounded, and we will be done if we can prove that \( \bigcap_{i \in I} C_i \) is closed and bounded as well. Call this intersection \( C \). By Corollary 13.11(a), we know that \( C \) is closed, since each \( C_i \) is closed and the intersection of any family of closed sets is closed. Now fix one of the sets \( C_i \); call it \( C_{i_0} \). (It doesn’t matter which one we fix, just that we fix one.) Then it is certainly true that \( C \subset C_{i_0} \); \( C \) is the intersection \( \bigcap_{i \in I} C_i \), so if \( x \in C \), we know \( x \in C_i \) for all \( i \in I \), and in particular, \( x \in C_{i_0} \). But \( C_{i_0} \) is compact and hence bounded, so by the lemma above, \( C \), being a subset of a bounded set, is bounded. Since \( C \) is both closed and bounded, the Heine-Borel theorem tells us \( C \) is compact.