(1) Using the definitions from lecture, prove that the intersection of infinitely many closed sets is a closed set. (If you reduce/modify this statement to one about open sets, then you should prove that statement about open sets and not just cite a previous result.)

Let \( \{ A_n \mid n \in \mathbb{N} \} \) be a family of closed sets. Then \( \forall n \in \mathbb{N}, \ B_n = \mathbb{R} - A_n \) is an open set (by def'n of "closed").

To prove that \( \bigcap_{n=1}^{\infty} A_n \) is closed, we need to show that \( \mathbb{R} - \bigcap_{n=1}^{\infty} A_n \) is open.

By DeMorgan's law, \( \mathbb{R} - \bigcap_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \). So let \( x \in \bigcup_{n=1}^{\infty} B_n \).

Then \( x \in B_k \) for some \( k \in \mathbb{N} \), and since \( B_k \) is open, there is some neighborhood \( N(x, \varepsilon) \subset B_k \subset \bigcup_{n=1}^{\infty} B_n \). Thus \( x \) is an interior point of \( \bigcup_{n=1}^{\infty} B_n \), so \( \bigcup_{n=1}^{\infty} B_n \) is open. \( \square \)

(2) Determine whether the set \( S = \{ 1 - \frac{1}{n} : n \in \mathbb{N} \} \subset \mathbb{R} \) is open, closed, both or neither. Justify your answer using any definitions or theorems from class.

**Neither.** To see that \( S \) is not open, we show \( 0 \in S \) is not an interior point of \( S \):

\( 0 \in S \) because \( 0 = 1 - \frac{1}{1} \). But any neighborhood \( N(0, \varepsilon) \) contains negative numbers, and \( 1 - \frac{1}{n} > 0 \) \( \forall n \in \mathbb{N} \).

To see that \( S \) is not closed, we show \( 1 \in \mathbb{R} - S \) is not an interior point of \( \mathbb{R} - S \):

\( 1 \in \mathbb{R} - S \) because \( \frac{1}{n} > 0 \) \( \forall n \in \mathbb{N} \), so \( 1 - \frac{1}{n} < 1 \) \( \forall n \in \mathbb{N} \). But for any neighborhood \( N(1, \varepsilon) \), by the Archimedean property, \( \exists N \in \mathbb{N} \) s.t. \( N > \frac{1}{\varepsilon} \), so \( \varepsilon > \frac{1}{N} \), so \( 1 - \varepsilon < 1 - \frac{1}{N} < 1 \).

Thus \( 1 - \frac{1}{N} \in N(1, \varepsilon) \), so \( N(1, \varepsilon) \notin \mathbb{R} - S \). \( \square \)