§ 2. Quantifiers

"Quantifiers"... quantify things. In logic we mostly care if a statement is true in all cases, [at least] one case, or never.

Being lazy, we use symbols

Existential Quantifiers

\( \exists \): there exists... (at least one)

\( \exists! \): there exists a unique (exactly one)

\( \forall \): there does not exist

Universal

\( \forall \): for all
Other notation

\[ \exists \text{ such that} \]

\[ p(x): \text{stmt whose truth value depends on value of } x. \]

\[ p(x) : x^2 - 1 = 0 \]

\[ p(0) \text{ false} \]

\[ p(1) \text{ true.} \]

Ex. Write these stmts in symbols:

For some \( x \), \( x^2 - 1 = 0 \).

\[ \exists x \in \mathbb{R} : x^2 - 1 = 0. \]

For every real number \( x > 0 \), there is a \( y \) s.t. \( y^3 = x \).

\[ \forall x > 0 \exists y \in \mathbb{R} : y^3 = x. \]

Every real \( \# \) has a cube root.

\[ \forall x \exists \text{cube root } y !: \forall x \exists y \in \mathbb{R} : y^3 = x. \]

For every \( \# \) there is a larger \( \#. \)

\[ \forall x \exists y \in \mathbb{R} : y > x. \]

There is a largest real number.

\[ \exists x \in \mathbb{R} : \forall y, x > y. \]
Two more:

If $x>1$, then $x^2>1$.

$(\forall x \in \mathbb{R}, \ x > 1 \implies x^2 > 1)$

There is no square root of $-2$ in $\mathbb{R}$.

$\forall x \in \mathbb{R}, \ x^2 \neq -2$.  (slang...)

$\forall x \in \mathbb{R}, \ x^2 \neq -2$.

⚠️ Negation of statements with quantifiers is tricky...

In words: the negation of "every day is sunny" isn't "every day is rainy"!

It's "At least one day is not sunny.

or "At least one day is rainy".

[Here ~ sunny = rainy]
Symbolically

- Negation of $\forall x, p(x)$ is $\exists x, \neg p(x)$.

i.e.

$\neg [\forall x, p(x)] \iff \exists x \exists 3 \neg p(x)$

- Also

$\neg [\exists 3 p(x)] \iff \forall x, \neg p(x)$

Ex: Negate these stmts:

(a) $\forall x, g(x) < 0$

$\exists x \exists 3 g(x) \geq 0$.

(b) $\exists x \exists f'(x) = 0$

$\forall x, f'(x) \neq 0$.

(c) $\forall 3 A \exists 3 B \exists 3 0 < 1x - n + 1 < 16$