§5: Basic Set Thy

Set Thy: Seems tedious at first, but **essential**. As you progress to higher level courses, the language of set thy replaces arithmetic!

**Def** A set is an unordered collection of objects called elements. Write \( x \in A \) to denote that \( x \) is an elt (or member) of \( A \).

If \( A \) has finitely many elt,

\[ |A| = \# \text{ of elt in } A \]

= cardinality of \( A \).

(Else \( A \) is infinite).
Ways to define sets

listing elts
A: \{1, 2, 3, 4, 5\}
B: \{0, \Delta, \Box\}

defining property
C = \{x \mid x > 0\} = \{x : x > 0\}
superset: \{x \geq 0\}

Notes 1 A "universal set" is often implied or assumed.
A: integers?  B: shapes?
C: real #'s?

Standard names
N: natural #'s = \{1, 2, 3, 4, \ldots\}
Z: integers = \{-2, -1, 0, 1, 2, \ldots\}
Q: rational #’s
R: real #’s
C: Complex #’s
F = finite field
P^a \setminus P^a elts.

(-4z3) = \{x \in \mathbb{R}\}
x \geq 1 \land x < 3 
\phi = \{\}\
Subsets: A is a subset of B, \( A \subseteq B \), if \( x \in A \Rightarrow x \in B \).

Ex: \( B = \{1, 2, 3, 9\} \).
- \( A = \{1, 3, 4, 2\} \subset B \) — non-proper
- \( A = \{1, 3\} \subset B \) — proper
- \( A = \{1, 2, 4, 5\} \) Not! (is not a subset)
- \( \emptyset \subset B \) — proper

A subset of B is **proper** if it doesn’t contain all the elts of B. i.e. \( C \) but \( \neq \)

**Notes**
1. To prove \( A = B \), must show \( A \subseteq B \) and \( B \subseteq A \) (e.g. \( \subseteq \) and \( \subseteq \))
2. Some books use \( C, \subseteq \) for proper, proper or equal. (Think \( <, \subseteq \))
   - Most use \( C \) for both.
   - Our book uses \( \subseteq \) for both.
Forming new sets from old.

Intersection \( A \cap B = \{ x \mid x \in A \land x \in B \} \)

Venn Diagrams:

Union \( A \cup B = \{ x \mid x \in A \lor x \in B \} \)

Complement \( \overline{A} = A^C = \{ x \mid \neg (x \in A) \} = \{ x \mid \neg A \} \)

Our book: if \( X \) is universal set, \( \overline{A} = A^C = X \setminus A \).

Set Difference \( A - B = A \setminus B = \{ x \mid x \in A, x \notin B \} \)

= complement of \( B \) in \( A \)
Ex In $\mathbb{N}$, $A = \text{even #s, } B = \{1, 3, 5, ..., 10\}$.

$A \cap B = \{2, 4, 6, 8, 10\}$
$A \cup B = \{1, 2, 3, ..., 10\}$
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\bar{A} = \text{odds}
A \triangle B = \{12, 14, 16, ..., 30\}
B \setminus A = \{11, 13, 15, 17, 19\}
A \cup \emptyset = A
B \cap \emptyset = \emptyset
$e

Ex Prove $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

\text{like: } x \cdot (y + z) = (x \cdot z) + (x \cdot y)$

\text{PF: } We must show $X \cap (Y \cup Z) \subseteq (X \cap Y) \cup (X \cap Z)$

Let $x$ be any elt in $X \cap (Y \cup Z)$. We want to show $x \in (X \cap Y) \cup (X \cap Z)$. 
$x \in X \cap (Y \cup \emptyset)$, so $x \in X$ and $Y \cup \emptyset$.

In particular, $x \in Y \cup \emptyset$ means $x \in Y$ or $x \in \emptyset$ (or both).

If $x \in Y$, then $x \in X \cap Y$ since it's in both $X$ and $Y$.

Similarly, if $x \in \emptyset$, then $x \in X \cap \emptyset$.

Hence $x \in X \cap Y$ or $x \in X \cap \emptyset$ (or both).

$\Rightarrow x \in (X \cap Y) \cup (X \cap \emptyset)$.

Thus $X \cap (Y \cup \emptyset) \subseteq (X \cap Y) \cup (X \cap \emptyset)$.

You try: $\subseteq$
Warmup: Prove $A \setminus B = (U \setminus B) \setminus (U \setminus A)$
where $U$ is the universal set.

Key: $A \setminus B \subseteq (U \setminus B) \setminus (U \setminus A)$, $(U \setminus B) \setminus (U \setminus A) \subseteq A \setminus B$.

Proof (No words — glibly bad!)

$x \in A \setminus B \iff x \in A$ and $x \notin B$.

$\iff x \in (U \setminus B)$ and $x \notin (U \setminus A)$

$\iff x \in (U \setminus B) \setminus (U \setminus A)$

Hence $\text{LHS} \subseteq \text{RHS}$.

Next, let $x \in (U \setminus B) \setminus (U \setminus A)$, which means $x \in U \setminus B$ and $x \notin U \setminus A$.

I.e. $x \notin B$ and $x \in A$.

Hence $x \in A \setminus B$, and $\text{RHS} \subseteq \text{LHS}$.

Hence shown both inclusions,
we see that the sets are equal.

Alternatively, could change each
$\iff$ in 1st half to $\iff$. \(\square\) \(\square\) \(\square\)
Indexed Sets

Often we use families of sets.

\[ E \times A_n = [-n, n], \ n \in \mathbb{N}. \]

\[ A_1 = [-1, 1] \quad A_{100} = (-100, 100) \quad n: \text{index} \]

\[ A_\infty = (-\infty, \infty) \]

\[ \mathbb{N}: \text{index set} \]

\[ \text{set of indices.} \]

We'll often use notation similar to

\[ \sum A_n = a_1 + a_2 + a_3 + \cdots + a_n \]

when dealing with indexed sets.

\[ \bigcup_{n=1}^{5} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \]

\[ = [-1, 1] \cup [-3, 3] \cup \cdots \cup [-5, 5] \]

\[ = [-5, 5] \]

To prove: Let \( x \in [-5, 5] \), show it's in \( \bigcup_{n=1}^{5} A_n \) and vice versa. (two inclusions).

Let \( x \in (-5, 5) = A_5 \). Since \( A_5 \) is a subset of \( A_1 \cup A_2 \cup \cdots \cup A_5 \), \( x \in \bigcup_{n=1}^{5} A_n \). 

Ex: $\bigcap_{n=1}^{\infty} A_n = [-4,1] \cap [-2,3] \cap \ldots = [-1,1]$

**Proof:** First let $x \in \bigcap_{n=1}^{\infty} A_n$, so $x \in A_n \forall n$. In particular, $x \in A_1 = [-1,1]$. Thus $x \in [-1,1]$ and $\bigcap_{n=1}^{\infty} A_n \subseteq [-1,1]$

Conversely, let $x \in [-1,1]$. Then $x \in [-n,n]$ for all $n \in \mathbb{N}$ i.e. $x \in A_n$ for all $n \in \mathbb{N}$ \[\Rightarrow x \in A_1 \cap A_2 \cap A_3 \cap \ldots = \bigcap_{n=1}^{\infty} A_n.\]
Thus $[-1,1] \subseteq \bigcap_{n=1}^{\infty} A_n.$

Hence \[\bigcap_{n=1}^{\infty} A_n = [-1,1].\]