§7. Functions

Contains a variety of def'ns, including "function" as a rel'n, i.e. subset of a Cartesian Prod.

Most important parts:
- $f: A \rightarrow B$ notation, domain, range, codomain.
- injective, surjective, bijective
- images and pre-images of sets, $f(c)$ and $f^{-1}(D)$.
- inverses and compositions.
§7. Functions

≤ Calc I, functions generally have one input \((x, t, \ldots)\) and one output \((y, f(t), g(x)) : \ y = \sin t\)

≥ Calc III, Linear Algebra we have a more general notation:

\[ f : A \rightarrow B \]

* Domain \(A\) inputs (set of all inputs)
  * codomain \(B\) outputs
  * target set \(B\)

\[ \text{Range of } f = \{ b \in B \mid b = f(a) \text{ for some } a \in A \} \]

⚠️ \(f\) must assign an output to each elt in its domain!

⚠️ In many books \(B\) (codomain) called the range.
Ex: \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \)
\( (x, y) \mapsto \frac{y+1}{x} \)
\[ z = f(x, y) \]
\[ z = \frac{y+1}{x} \]

domain: \( \{ (x, y) \in \mathbb{R}^2 \mid x \neq 0 \} \)
\( \{ (x, y) \in \mathbb{R}^2 \mid x \neq 0 \} \)
codomain = \( \mathbb{R} \), range \( f \) =

⚠️ Should write
\( f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \).
\( D = \{ (x, y) \mid x \neq 0 \} \)

\( f: D \subseteq \mathbb{R} \rightarrow \mathbb{R} \)
\( x \mapsto \sqrt{4-x^2} \)

\( D = [-2, 2] \)
range \( f \) = \( [0, 3] \)

so codomain \( \neq \) range.

but I could write/redefine \( f: D \subseteq \mathbb{R} \rightarrow [0, 3] \)
\( x \mapsto \sqrt{4-x^2} \)
This book is even more general...

**Def** A function \( f \) between sets \( A \) and \( B \) is a nonempty subset \(^*\) of \( A \times B \) s.t. if \( (a, b) \in f \) and \( (a, b') \in f \), \( b = b' \); i.e. a reln.

Instead of giving a formula to define \( f \), this method just lists all inputs and their associated outputs.

The extra condition ensures that any input \( a \) has just one output.

**Ex:** \( f: \{1, 2, 3, 4, 5\} \rightarrow \mathbb{N} \quad f(n) = n^2 + 1 \)

\( f = \{(1, 2), (2, 5), (3, 10), (4, 17)\} \)
In this method, a fn $f : A \rightarrow B$ need not have domain $A$.

$A = \{1, 2, 3\}$, $B = \mathbb{N}$,

$f = \{1, 102, 13, 423\} - 2 \notin A$, $2 \notin \text{dom } f$.

If we write $f : A \rightarrow B$, it implies $\text{dom } f = A$.

There are two common ways to visualize fn's:

1. Graphs: when inputs/outputs are #’s
   inputs usually horiz. coords
   outputs usually vert. coords.

2. "Blob" Diagrams.

$\text{dom } f = A$
Ex  $f: T \rightarrow P$

$T = \text{tomatoes}$

$P = \text{professors}$

**Glossary of Terms for $f: A \rightarrow B$**

- *surjective*, *onto*: everything in $B$ gets hit.

$f: A \rightarrow B$ is onto if $\forall b \in B$ there exists $a \in A$ such that $f(a) = b$. 

- range
- domain
injective, one to one, 1:1

Each output is hit by only one input.

If \( f(x_1) = f(x_2) \) then \( x_1 = x_2 \).

Or (contra) \( x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \).

bijective

both injective and surjective.

Each \( b \in B \) is hit by exactly one input \( a \in A \).

"relabeling"

inverse fn of \( f \) is a fn \( f^{-1}: B \rightarrow A \)

s.t. \( f^{-1}(f(a)) = a \ \forall a \in A \).

\( f(f^{-1}(b)) = b \ \forall b \in B \).

i.e. \( f^{-1} \circ f = \text{id}_A: A \rightarrow A \), and

\( f \circ f^{-1} = \text{id}_B: B \rightarrow B \), where

\( f^{-1} \)

is the inverse function of \( f \).
Set defn of $f^{-1} : B \to A$

$$f^{-1} = \{(b, a) \mid f(a) = b\} \text{ and if } (b, a) \text{ ad } (b, a') \text{ then } a = a'$$

Existence of $f^{-1}$ depends on $f$ being bijective.

($f$ is bijection)
Functions acting on subsets \( f: A \to B \)

Let \( C \subseteq A \), \( D \subseteq B \).

\[
f(C) = \{ (x,y) \in f \mid x \in C \} \quad (44)
\]

i.e. \( f(C) = \{ f(x) \mid x \in C \} \).

"image of \( C \) under \( f \)."

\[
f^{-1}(D) = \{ a \in A \mid f(a) \in D \}
\]

"preimage of \( D \) under \( f \)."

Example: \( f: \mathbb{R} \to \mathbb{R} \), \( f(x) = x^2 \).

\[
f([0,2)) = (0,4)
\]

\[
f^{-1}((1,4]) = (1,2] \cup [-2,-1).
\]

\[
f([-1]) = \{ f(a) \mid a \in [-1] \} = \{ 1, 9, 16, ... \}
\]

\[
f^{-1}([-9,-4]) = \emptyset
\]
**Composition** \( g \circ f(x) = g(f(x)) \)

In pictures, if \( f: A \to B, \ g: B \to C, \)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g \circ f} \\
C & \xrightarrow{=} & C
\end{array}
\]

sof. (watch the dir'n!)

**READ THE BOOK CAREFULLY AND ASK QUESTIONS.**

**Many things like...**

- **Thm 7.14(a)** \( f, g \) surjective \( \Rightarrow \) \( g \circ f \) surj.
- **Thm 7.15(c)** \( f(c_1 \cap c_2) \subseteq f(c_1) \cap f(c_2) \)
- **Thm 7.17(a)** \( f \) inj. \( C \subseteq A \Rightarrow f^{-1}[f(C)] = C \)

⚠️ Rather than memorize everything, practice this type of proof so you can "do it on the fly!"
Thm 7.19 (a). Let \( f: A \to B \), \( g: B \to C \) be surjective. Then \( g \circ f \) is surj.

Note: \( g \circ f: A \to C \).

**Pf:** \( g \circ f: A \to C \), so to prove \( g \circ f \) is surjective we let \( c \in C \), want to show \( \exists a \in A \) such that \( g \circ f(a) = g(f(a)) = c \).

\( g: B \to C \) is surjective, so \( \exists b \in B \) s.t. \( g(b) = c \).

\( f: A \to B \) is surj, so \( \exists a \in A \) s.t. \( f(a) = b \).

Then \( g \circ f(a) = g(f(a)) = g(b) = c \).
Theorem 7.15(c) \[ f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2) \]

**Proof:** Let \( x \in C_1 \cap C_2 \). Since \( x \in C_1 \), \( f(x) \in f(C_1) \). Since \( x \in C_2 \), we also have \( f(x) \in f(C_2) \).

Thus \( f(x) \in f(C_1) \cap f(C_2) \).

\[ \Rightarrow f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2). \]