§12 Reminder

Def \( m = \sup S \) (SSR) iff

(a) \( m \geq s \) \( \forall s \in S \)
(b) \( m < m' \Rightarrow \exists s \in S \) s.t. \( m' < s \).

⚠️ In practice, we prove (b) by letting \( \varepsilon > 0 \) (presumably small), setting \( m' = m - \varepsilon \).

Today: §13 Topology ≠ Topography

Many topics/defs here:

* 1. neighborhoods
* 2. interior, boundary pts.
* 3. open, closed sets in \( \mathbb{R} \)
* 4. accumulation pts
* 5. closure of a set.
§13. Topology of \( \mathbb{R} \)

In this context, "topology" refers to "open and closed sets."

With sequences, limits, we talk about "points close to \( x \)." In this section we start to give that idea a careful def.

\[ \text{Def Let } x \in \mathbb{R}, \varepsilon > 0. \text{ Then } \]
\[ B_\varepsilon(x) = N(x;\varepsilon) = \{ y \in \mathbb{R} \mid |x-y| < \varepsilon \} \]
\[ \text{is a neighborhood (nbhd) or } \varepsilon \text{-nbhd of } x, \text{ with radius } \varepsilon. \]
\[ N^d(x;\varepsilon) = \{ y \mid 0 < |x-y| < \varepsilon \} = (x-\varepsilon, x) \cup (x, x+\varepsilon) \]

is deleted nbhd of \( x \).

\[ \text{Ex } \]
\[ N(2;1) = (1,3). \]
\[ N^d(2;1) = (1,2) \cup (2,3) \]
Def \( x \in \mathbb{S} \in \mathbb{R} \) is an interior point of \( S \) if \( \exists \) nbhd \( N \) of \( x \) in \( S \): \( N \subseteq S \).

\[ E \in (0,5) \]

\[ \begin{array}{c}
N(3;1) = (2,4) \cap (0,5) \checkmark \\
N(3;3) = (1,5) \cap (0,5) \checkmark \\
N(3;4) = (-1,7) \notin (0,5) \times
\end{array} \]

If every nbhd of \( x \) contains pts in \( S \) \((N_nS \neq \emptyset)\) and also contains pts not in \( S \) \((N_n(M \setminus S) \neq \emptyset)\), then \( x \) is a boundary point of \( S \).

\[ E \in [0,4] \]

\[ \forall E > 0, N(x;E) \text{ will contain pts in } [0,4] \text{ and pts } > 4 \text{ (hence in } \mathbb{R} \setminus [0,4] ) \]
Ex $S = \{0, 2, 4\}$

No int. pts!
Bdy pts: $0, 2, 4$.

\[0 \quad 2 \quad 4 \quad \rightarrow \mathbb{R}\]

\[\forall \epsilon > 0, \ N(2; \epsilon) \cap S \Rightarrow 2 \text{ not interior pt.}\]
\[N(2; \epsilon) \text{ contains } 2, \text{ it's not in } S, \text{ does contain } 2 \in S.\]

\[\Rightarrow \text{ a bdy pt.}\]

Ex $T = [0, 1)$

Choose $x \in \mathbb{R}$, $0 < x < 1$.

Compute $|x - 0|$, $|x - 1|$.

Choose $\delta$ to be the smaller of the two.

$\Rightarrow N(x; \delta) \subseteq [0, 1)$

Any $N(0; \epsilon)$ will contain pts in, out of $[0, 1) \Rightarrow 0$ is bdy pt.

Same for 1
(bdy pts need not be in the set.)
Def: $S \subseteq \mathbb{R}$ is open if every pt in $S$ is an interior pt, i.e:

$$\forall x \in S \exists \varepsilon > 0 \exists B_N(x; \varepsilon) \subseteq S$$

Examples

1. Any interval $(a, b) = \{ a < x < b \}$ is open.

Let $x \in (a, b)$, choose

$$\varepsilon = \min \{ b-x, x-a \}$$

Then $B_N(x; \varepsilon) \subseteq (a, b)$

2. Since $B_N(x; \varepsilon) = (x-\varepsilon, x+\varepsilon)$, neighborhoods are open. 
   Can Refer to "open disk."
3. \( \mathbb{R} \) is open.
   
   Let \( x \in \mathbb{R} \). For any \( \varepsilon > 0 \),
   
   \( N(x; \varepsilon) \subseteq \mathbb{R} \). Done.

4. \( \emptyset \subseteq \mathbb{R} \) is open.
   
   If \( x \in \emptyset \), then \( x \) is interior pt.
   
   always FALSE
   
   for \( S = \emptyset \), so implication \( \Rightarrow \) TRUE.

5. \( S = (0,1) \cup (4,5) \)
   
   \( x \in S \Rightarrow x \in (0,1) \) or \( x \in (4,5) \).
   
   If \( x \in (0,1) \), \( (0,1) \) is an open set,
   
   so \( \exists \varepsilon \) s.t.
   
   \( N(x; \varepsilon) \subseteq (0,1) \subseteq S \)
   
   Similar for \( x \in (4,5) \) \( \Rightarrow \) \( S \) open.

6. \( S = (1,6) \cap (2,9) \)
   
   \( x \in S \Rightarrow x \in (1,6) \) and \( x \in (2,9) \),
   
   both of which are open.
   
   \( \exists \varepsilon_1, \varepsilon_2 > 0 \) \in
   
   \( N(x; \varepsilon_1) \subseteq (1,6), \)
   
   \( N(x; \varepsilon_2) \subseteq (2,9) \).
Here generally,

Thm 13.10

(a) any union of open sets is open.
(b) \( \cap \) of finitely many open sets is open.

**Proof:**

(a) Suppose \( A_j \) is open for all \( j \in J \). (\( J = \mathbb{N} \)? \( \{1, 2\} \)? etc.)

Let \( x \in \bigcup A_j \Rightarrow x \in A_n \), some \( j \in J \).

\( A_n \) is open by assumption,

so \( \exists \varepsilon > 0 \) s.t. \( N(x; \varepsilon) \subseteq A_n \).

Since \( A_n \cap \bigcup A_j \), we also have \( N(x; \varepsilon) \subseteq \bigcup A_j \), \( j \in J \).
PF (W): Suppose \( A_1, \ldots, A_n \) are open.

Let \( x \in A_1 \cap \cdots \cap A_n \Rightarrow x \in A_i \)

\[ \exists \varepsilon_i > 0, \varepsilon_2 > 0, \ldots, \varepsilon_n > 0 \]

\( \text{s.t.} \quad N(x; \varepsilon_i) \subseteq A_i, \quad N(x; \varepsilon_2) \subseteq A_2, \ldots, \quad N(x; \varepsilon_n) \subseteq A_n \)

If \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \} \)

then \( N(x; \varepsilon) \subseteq A_1 \cap \cdots \cap A_n \).

\( \Rightarrow A_1 \cap \cdots \cap A_n \text{ open.} \)

⚠ Finitely many necessary here

by \( c \min \{ \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \} \) might not exist.

\( \exists x \bigcap_{n \in \mathbb{N}} \left( -\frac{1}{n}, \frac{1}{n} \right) = \emptyset \) not an open set.
Def: \( S \subseteq \mathbb{R} \) is closed if \( \mathbb{R} \setminus S = S^c \) is open.

⚠️ closed, open \textbf{NOT} opposites:

"not open" ≠ closed!

Think: \([0,1]\)

\section*{Examples}

1. \([0,1] = S. \quad \mathbb{R} \setminus S = S^c = (-\infty,0) \cup (1,\infty)\)
   open, so
   \( S \) is closed.

2. \((1,2]. \quad \mathbb{R} \setminus (1,2] = (-\infty,1] \cup (2,\infty)\)
   \( \uparrow \) not open
   \( \uparrow \) open

3. \( \mathbb{R} \)
   \( \mathbb{R} \setminus \mathbb{R} = \emptyset \)
   \( \emptyset \) is open \( \Rightarrow \) \( \mathbb{R} \) is closed.

\( \mathbb{R} \) is "clopen".
Different Characterization:

Thm: $S \subseteq \mathbb{R}$ closed if it contains all its bdy points.

⚠️ That’s the def in book. S Our def then given as a Thm.

Interiors and boundaries clearly important, so we name these sets:

Def: $\text{int } S = \text{set of interior pts}$

$\text{bd } S = \text{set of bdy pts}$.

Above def's, thms can be written:

$S$ open iff $S = \text{int } S$.

$S$ closed iff $\text{bd } S \subseteq S$. 
Cool Stuff

Any of finitely many open sets is open, but any of infinitely many can be closed!

\[ \bigcap_{n=1}^{\infty} A_n = \emptyset \text{ which is not an open set, b/c any } \mathbb{N}(0;\varepsilon) \text{ will include pts not in the intersection.} \]

\[ \bigcup_{n=1}^{\infty} A_n = \mathbb{R} \text{ which is open or } \mathbb{R} \text{ closed.} \]

Similarly, any of finitely many closed sets is closed, but any of infinitely many can be open!

\[ \bigcup_{n=1}^{\infty} B_n = \mathbb{R} \text{ open (and clopen)} \]

\[ U \left[ \frac{1}{n}, 2 - \frac{1}{n} \right] \cdots \cup (0, 2) \text{ open but not closed.} \]
A hybrid of interior, bdg pts:

**Def** \( x \in \mathbb{R} \) is an accumulation pt of \( S \) if \( \forall \varepsilon > 0, \ N^*(x;\varepsilon) \cap S \neq \emptyset \).

\( x \) need not be in \( S \) to be an acc. pt of \( S \).

- \( \exists x \in \mathbb{R} \) s.t. \((x-\varepsilon, x) \cup (x, x+\varepsilon) \)

\( x = \frac{7}{10} \) is acc pt of \( S \).

\( 0 \) is acc pt. \( 1 \) too.

Set of acc pts: \( S' = [0,1] \)

IN \& \( \mathbb{R} \)

Any \( n \in \mathbb{N} \) is bdg pt of \( \mathbb{N} \) (\( \mathbb{N}(n;\varepsilon) \) will include \( n \) and it's not in \( \mathbb{N} \)).

But no \( n \in \mathbb{N} \) is an acc. pt.

(Set of acc pts is \( \emptyset \)).

\( \mathbb{N} \) is an isolated pt.
Def: closure of $S = \text{cl } S = S \cup S'$

Ex: $\text{cl } (0,1) = (0,1) \cup \{0,1\} = [0,1]$

Thm: Let $S \subseteq \mathbb{R}$.

(a) $S$ closed $\iff S' \subseteq S$.
(b) $\text{cl } S$ is closed.
(c) $S$ closed iff $S = \text{cl } S$.
(d) $\text{cl } S = S \cup \text{bd } S$.

\[ S \cup S' = \text{cl } S = S \cup \text{bd } S \]

\[ \text{def} \quad (d) \]

$\Rightarrow S' = \text{bd } S$

Think: $[0,1]$, where $S' = (0,1]$ and $\text{bd } S = \{0,1\}$.