§ 18 Monotone and Cauchy Sequences

Or: More ways to prove a sequence converges without using the E-N def. Concepts

1. Monotone convergence Th.
2. Applications to certain recursive sequences
3. Cauchy Sequences

Def: \((s_n)\) is increasing if \(s_n \leq s_{n+1} \ \forall n\)

decreasing if \(s_n \geq s_{n+1} \ \forall n\)

With \(<, >\) instead of \(\leq, \geq\), we'd say strictly incr's, decr's.

Allowing for equality gives us monotonically incr's, decr's. (default)

\((s_n)\) monotone \((\rightarrow)\) (mon'th) incr's or decr's
Examples of seq's which are...

**increasing** \( s_n = n \) (strictly)

**decreasing** \( t_n = -n \) (strictly)
\( r_n = \frac{1}{n} \)

**monotonic increasing, not strictly** \((a_n) = (1, 1, 3, 3, 3, 4, 4, \ldots)\)
\(= (\lfloor \frac{1}{2} + \frac{n}{2} \rfloor), \; \lfloor \cdot \rfloor = \text{floor} \)

**both increasing and decreasing** \((b_n) = (1, 1, 1, 1, \ldots)\)

Three ways to show \((a_n)\) is increasing:

1. **Directly** \( a_{n+1} \geq a_n \) \(\forall n\) via algebra:
   \[
   \frac{a_{n+1}}{a_n} \geq 1 \quad \text{(if } a_n > 0 \text{, } \forall n)\]
   \(a_{n+1} - a_n \geq 0\)

2. **Induction.** \( a_2 \geq a_1 \). Then assume \( a_k \geq a_{k-1} \) show \( a_{k+1} \geq a_k \)

3. **Calculating.** \( a_n = f(n) \) — “nice” \( f: \mathbb{N}, f' > 0 \).
Thm 18.3 Monotone Convergence Thm

A monotone seq is convergent if its **bld**.

Why? Draw a picture!

![Graph showing a monotone sequence]

**VERY USEFUL** because we can show a seq. converges without resorting to ε's, N's, etc.

**Ex** Show \((a_n) = (1 - \frac{1}{n})\) converges.

(bdd below by 0)

bdd above \(1 - \frac{1}{n} = \frac{n-1}{n} < 1\) \(\forall n\).

\[a_n = a_{n+1} = (1 - \frac{1}{n+1}) = (1 - \frac{1}{n})\]

\[= \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \geq 0\]

\(\Rightarrow\) \(a_n\) converges by M.C.T.
Ex 18.4 Let \( S_1 = 1 \), \( S_{n+1} = \sqrt{1 + S_n} \)

\[ S_n = \left( 1, \sqrt{2}, \sqrt{1 + \sqrt{2}}, \sqrt{1 + \sqrt{1 + \sqrt{2}}}, \ldots \right) \]

**Claim:** \( 2 \) is an upper bd of \( \{S_n: n \in \mathbb{N}\} \)

*By induction*

**Base:** \( S_1 < 2 \).

Assume \( S_k < 2 \), prove for \( S_{k+1} \):

\[ S_{k+1} = \sqrt{1 + S_k} < \sqrt{1 + 2} = \sqrt{3} < 2. \]

**Claim:** \( S_n \) is incr.

*By induction:*

**Base:** \( S_0 \leq S_0 \).

Assume \( S_{k-1} \leq S_k \).

\[ S_{k+1} = \sqrt{1 + S_k} \leq \sqrt{1 + S_{k+1}} = S_{k+1}. \]

\( \Rightarrow \) By MCT, \( S_n \) converges.

**To what?** For any conv. seq. \( \{a_n\} \)

\[ s = \lim S_n = \lim S_{n+1} = S \]

Here \( \lim S_n = \lim \sqrt{1 + S_n} \)

\[ S = \sqrt{1 + S} \]

\[ S^2 = 1 + S \quad \therefore \quad S = \frac{1 + \sqrt{5}}{2} \]
**Fibonacci Sequence**

\[
\begin{align*}
&f_1 = 1 \\
&f_2 = 1 \\
&f_n = f_{n-1} + f_{n-2}
\end{align*}
\]

\((f_n) = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)\)

\[
\frac{f_{n+1}}{f_n} = 1.618 \quad (?)
\]

\(\frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2}\)

4 the golden ratio

\(\approx 1.618\)

Diagram of the Fibonacci sequence in geometric form.
Proof of Monotone Convergence Theorem

(Incresg, bdd above case)

Let \( (s_n) \) be incresg, bdd above - i.e. set \( \{s_n : n \in \mathbb{N}\} \) bdd above

\( \Rightarrow \exists \ s = \sup \{s_n\} \). We claim \( s_n \to s \).

Need to show:

\[
\forall \varepsilon > 0 \ \exists \ N \in \mathbb{N} \ni n > N \Rightarrow |s_n - s| < \varepsilon
\]

Given \( \varepsilon > 0 \), \( s - \varepsilon \) not an upper bound. So \( \exists \ s_N \ s.t.

\[s - \varepsilon < s_N < s\]  

But \( s_n \) incresg \( \Rightarrow \forall \ n > N,

\[s - \varepsilon < s_n < s\]

\[\Rightarrow |s_n - s| < \varepsilon \forall n > N.\]
Cauchy Sequences

So far we've described "convergent as elts of seqince (eventually) bunching up next to a limit. Another way to characterize this:

**Def:** A seq. \((s_n)\) of real #s is a **Cauchy Sequence** if

\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } n, m > N \implies |s_n - s_m| < \varepsilon.
\]

Note this means elts in seq are bunching up together—close to each other, not some limit \(s\).
Ex \( S_n = \frac{(-1)^n}{n} \)

If \( \varepsilon = 1 \), how far down seq do we have to go to ensure \( |S_n - S_m| < 1 \) ? \((n, m > 1)\)

\[\forall \varepsilon > 0, \exists N \text{ s.t. } n, m \in \mathbb{N}, \]
\[n, m > N \Rightarrow \frac{\varepsilon}{3} < \frac{1}{n} \]
\[\frac{1}{n} < \frac{\varepsilon}{3} \]

Then, \( n, m > N \Rightarrow \)
\[|S_n - S_m| = \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \leq \left| \frac{(-1)^n}{n} \right| + \left| \frac{(-1)^m}{m} \right| = \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \]

Ex \( t_n = (-1)^n = (-1, 1, -1, 1, -1, 1, \ldots) \)

For \( \varepsilon = \frac{1}{2}, \nexists N \text{ s.t. } n, m > N \)
\[\Rightarrow |S_n - S_m| < \frac{1}{2} \]

(Could always have \( |1 - 1 - 1| = 2 > \frac{1}{2} \))
Lemma 18.10 \((s_n)\) converges \(\leftrightarrow\) (Cauchy)

**Proof:**

\(\Rightarrow\) not in this course.

\(\Rightarrow\) Suppose \(s_n \to s\), let \(\varepsilon > 0\) be given. Must show \(\exists N\) s.t. \(n, m > N\),

\[ |s_n - s_m| < \varepsilon. \]

We know \(\exists N\) s.t. \(n > N\)

\[ |s_n - s| < \varepsilon/2. \]

Then \(n, m > N\) gives

\[ |s_n - s_m| = |s_n - s - (s_m - s)| \leq |s_n - s| + |s_m - s| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]
AUGUSTIN CAUCHY
1789-1857

\[ f(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(z)}{z-a} \, dz \]

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