§18 Monotone and Cauchy Sequences

Or: More ways to prove a sequence converges without using the ε-N def$^{\underline{n}}$. Concepts

★★ 1. Monotone convergence Thm

★★ 2. Applications to certain recursive sequences

★ 3. Cauchy Sequences

Def (s_n) is increasing if $s_n \le s_{n+1}$ $\forall n$

decreasing if $s_n \ge s_{n+1}$ $\forall n$

With $<, >$ instead of $\le, \ge$, we'd say strictly incr'g, decr'g.

Allowing for equality gives us monotonically incr'g, decr'g. (Default)

(s_n) monotone $\Leftrightarrow$ (mon'g) incr'g or decr'g

Examples of seq's which are...

incr'g $s_n = n$ (strictly)

decr'g $\left.\begin{array}{l} t_n = -n \\ r_n = 1/n \end{array}\right\}$ (strictly)

non. incr'g, not strictly $(a_n) = (1,1,2,2,3,3,4,4,\dots)$
$= \left(\left\lfloor \frac{1}{2} + \frac{n}{2} \right\rfloor\right)$, $\lfloor\ \rfloor$ = floo

both incr'g and decr'g $(b_n) = (1,1,1,1,\dots)$

Three ways to show (a_n) is incr'g:

1. Directly $a_{n+1} \geq a_n \ \forall n$ via algebr
   or $\frac{a_{n+1}}{a_n} \geq 1$ (if $a_n > 0\ \forall n$)
   $a_{n+1} - a_n \geq 0$
2. Induction. ($a_2 \geq a_1$. Then assume $a_k \geq a_{k-1}$, show $a_{k+1} \geq a_k$).
3. Calculus.
   $a_n = f(n)$, "nice" fn f, $f' > 0$.

Thm 18.3 Monotone Convergence Thm

A monotone seq is convergent $\iff$ it's bdd.

Why? Draw a picture!

VERY USEFUL because we can show a seq. converges *without* resorting to ε's, N's, etc.

Ex Show $(a_n) = (1 - 1/n)$ converges.

(bdd below by 0)

bdd above $1 - \frac{1}{n} = \frac{n-1}{n} < 1 \quad \forall n.$

incr'g $a_{n+1} - a_n = \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right)$

$= \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)}$

$= \frac{1}{n(n+1)} \geq 0.$

$\Rightarrow a_n$ conv's by M.C.T.

Ex 18.4 Let $s_1 = 1$, $s_{n+1} = \sqrt{1+s_n}$

$s_n = \left(1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \ldots\right)$

1.414   1.55...   1.598....

Claim: 2 is an upper bd of $\{s_n : n \in \mathbb{N}\}$

Pf by induction Base: $s_1 < 2$.

Assume $s_k < 2$, prove for s_{k+1}:

$$s_{k+1} = \sqrt{1+s_k} < \sqrt{1+2} = \sqrt{3} < 2.$$

Claim: s_n is incr'g.

Pf by induction: Base: $s_1 \le s_2$. ✓.

Assume $s_{k-1} \le s_k$.

$$s_{k+1} = \sqrt{1+s_k} \geq \sqrt{1+s_{k-1}} = s_k.$$

$\Rightarrow$ By MCT, s_n converges.

To what? For any conv. seq,

$$s = \lim s_n = \lim s_{n+1} = s$$

Here $\lim s_n = \lim \sqrt{1+s_n}$

$$s = \sqrt{1+s}$$

$$s^2 = 1+s \quad \ldots\ldots \quad s = \frac{1+\sqrt{5}}{2}$$

Fibonacci Sequence

$f_1 = 1$

$f_2 = 1$

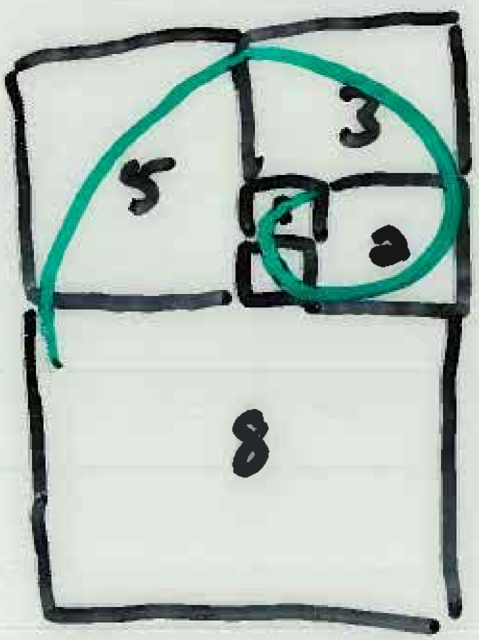
$$f_n = f_{n-1} + f_{n-2}$$

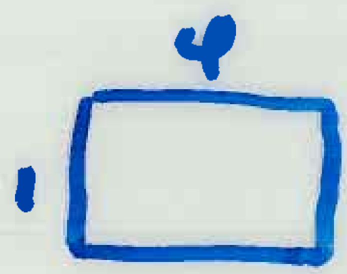
$$(f_n) = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$$

$$\frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}^{(?)}$$

φ the golden ratio

$\approx 1.6\ldots$





Proof of Monotone Conv Thm

(Incr'g, bdd above case)

Let (s_n) be incr'g, bdd above – i.e. set $\{s_n : n \in \mathbb{N}\}$ bdd above $\Rightarrow \exists\ s = \sup\{s_n\}$. We claim $s_n \to s$.

Need to show:

$\forall \varepsilon > 0\ \exists N \ni$

$n > N \Rightarrow |s_n - s| < \varepsilon$

Given $\varepsilon > 0$, $s - \varepsilon$ <u>not</u> an upper bound. So $\exists\ s_N$ s.t.

$$s - \varepsilon < s_N < s.$$

But s_n incr'g $\Rightarrow \forall\ n > N$,

$$\underline{s - \varepsilon < s_n < s.}$$

$$\Rightarrow |s_n - s| < \varepsilon \quad \forall n > N.$$

Cauchy Sequences

So far we've described "convergence" as elts of seq'nce (eventually) bunching up next to a limit. $\exists$ another way to characterize this:

Def A seq. (s_n) of real #'s is a *Cauchy Sequence* if $\forall \varepsilon > 0\ \exists N$ s.t. $n, m > N$ $\Rightarrow |s_n - s_m| < \varepsilon$.

Note this means elts in seq are bunching up together — close to each other, not some limit s.

Ex $s_n = \dfrac{(-1)^n}{n}$

If $\varepsilon = 1$, how far down seq do we have to go to ensure $|s_n - s_m| < 1$? $(n, m > 1)$

$\forall \varepsilon > 0, \exists N$ s.t. $n, m \in \mathbb{N}$,

$n, m > N \Rightarrow \dfrac{1}{n} < \varepsilon/2$

$\dfrac{1}{m} < \varepsilon/2.$

Then, $n, m > N \Rightarrow$

$$|s_n - s_m| = \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \leq \left| \frac{(-1)^n}{n} \right| + \left| \frac{(-1)^m}{m} \right|$$

$$= \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Ex $t_n = (-1)^n = (-1, 1, -1, 1, -1, 1, \ldots)$

For $\varepsilon = 1/2$, $\nexists N$ s.t. $n, m > N$

$\Rightarrow |s_n - s_m| < 1/2.$

(Could always have $|-1 - 1| = 2 > 1/2$)

Why do we care?

Lemma 18.10 (s_n) converges $\iff$ (s_n) Cauchy

Pf: $\Leftarrow$ not in this course.

$\Rightarrow$ Suppose $s_n \to s$, let $\varepsilon > 0$ be given. Must show $\exists N$ s.t. $n, m > N$,

$$|s_n - s_m| < \varepsilon.$$

We know $\exists N$ s.t. $n > N$

$$|s_n - s| < \varepsilon/2.$$

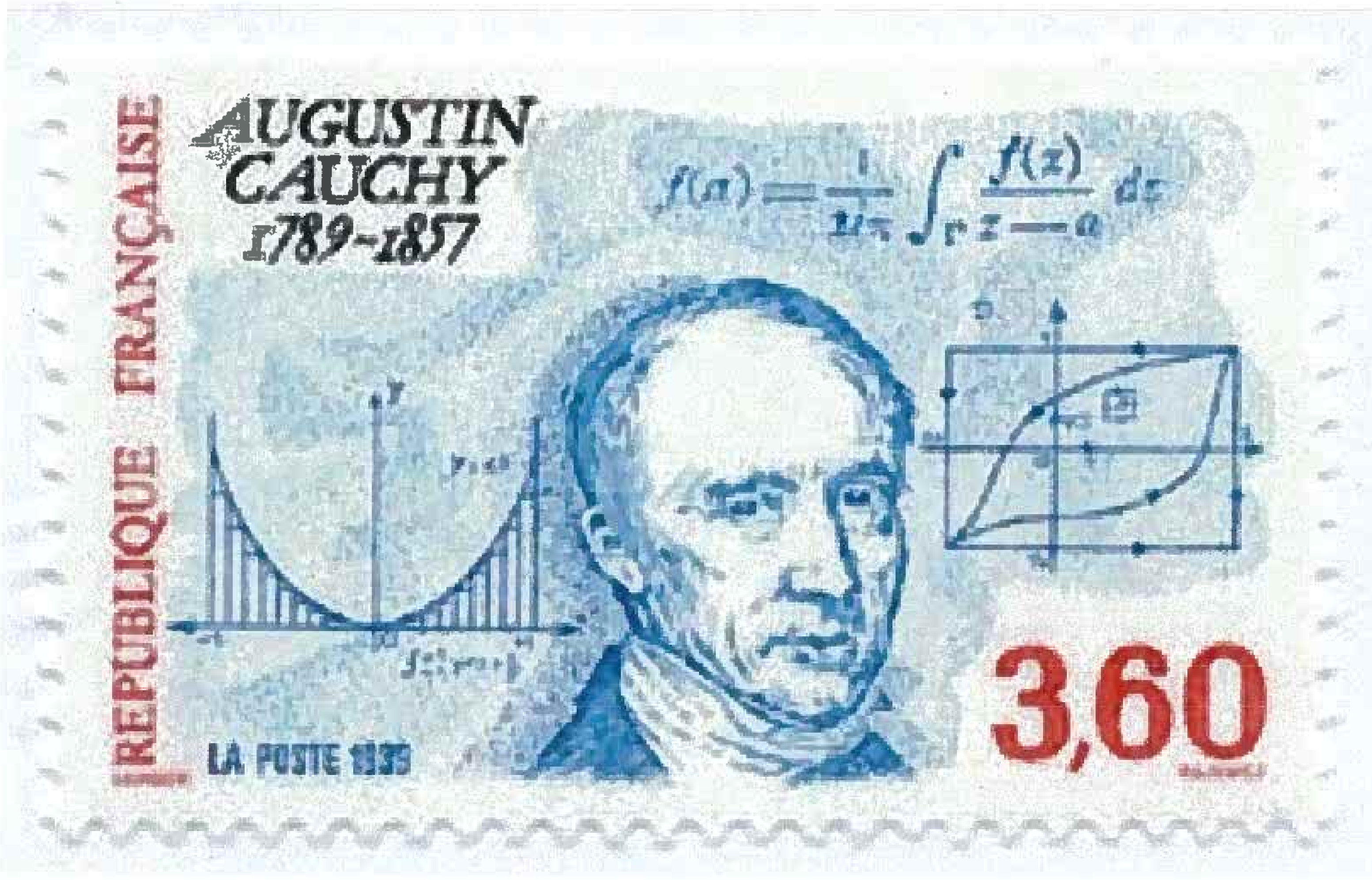
Then $n, m > N$ gives

$$|s_n - s_m| = |s_n - s - (s_m - s)|$$

$-s - (-s) = 0.$

$$\leq |s_n - s| + |s_m - s|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$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