Due Dates / Schedule

Th 11/29: last writing quiz

Tu 12/4: Takehome #3 (no skills)

Th 12/6: Exam #3

Tu 12/11: Skills

Th 12/13: Rewrites of TH #3
(in boxes by noon)

Today: Convergence Tests

Key: Can we analyze a_n's instead of $S_n = a_1 + \cdots + a_n$ to figure out if $\sum a_n$ conv's?

Alt. series test
ratio test, root test
comparison test, integral test.

§ 33 Convergence Tests

A basic test of divergence comes from Thm 32.5, $\sum a_n$ converges $\Rightarrow a_n \to 0$.

Contrapositive $a_n \not\to 0 \Rightarrow \sum a_n$ diverges

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!}{(n-1)!}$

$$a_n = \frac{(n+1)!}{(n-1)!} = \frac{(n+1)n(n-1)(n-2)\cdots 2\cdot 1}{(n-1)(n-2)\cdots 2\cdot 1}$$

$a_n = n^2+n \longrightarrow +\infty$, not 0

$\Rightarrow \sum a_n$ diverges!

★ This test (relatively) simple, so worth doing. But usually $a_n \to 0$ and we get no information; proceed to other tests.

<u>Thm 33.1</u> (Comparison Test). $0 \le a_n \le b_n \ \forall n$

(a) If $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

(b) $\sum a_n = +\infty \Rightarrow \sum b_n = +\infty$.

⚠ Switched roles of a_n, b_n from bk.

<u>Pf</u>: Let $\left.\begin{array}{l} s_n = a_1 + a_2 + \cdots + a_n \\ t_n = b_1 + b_2 + \cdots + b_n \end{array}\right\} s_n \le t_n$

(a) $a_n \ge 0 \Rightarrow s_n$ increasing

By earlier sections,

$s_n \le t_n \Rightarrow \lim s_n \le \lim t_n = t$

$\Rightarrow s_n$ <u>bounded</u> $\Rightarrow$ MCT

s_n converges $\Leftrightarrow \sum a_n$ converges.

(b) $s_n \to +\infty$, $s_n \le t_n \Rightarrow$

$t_n \to +\infty$.

Key to using Comparison Test:

(a) compare to a known series.

(b) make sure the direction of comparison is useful, i.e.

Ex $0 \leq \frac{1}{n^2} \leq \frac{1}{n} \ \forall n$ so

$$0 \leq \sum \frac{1}{n^2} \leq \sum \frac{1}{n} = +\infty$$

So comp. test tells us <u>nothing</u> about $\frac{1}{n^2}$ here.

Ex $\sum \frac{1}{n(n+1)} = 1$ (telescoping)

$$\forall n, \ 0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}$$

$$\Rightarrow 0 \leq \underbrace{\sum \frac{1}{n(n+1)}}_{1} \leq \sum \frac{1}{n^2}$$

<u>Not</u> a useful comparison.

$\exists$ Useful comparisons!

Ex $\sum \frac{1}{(n+1)^2}$ $\quad 0 \le \frac{1}{(n+1)(n+1)} \le \frac{1}{n(n+1)}$ $\forall n$

and $\sum \frac{1}{n(n+1)}$ converges to 1

$\Rightarrow \sum \frac{1}{(n+1)^2}$ converges by Comp. test.

To what?

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$= \left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) - 1$$

$$= \frac{\pi^2}{6} - 1$$ by last time (no proof).

Alt., let $u = n+1$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{u=2}^{\infty} \frac{1}{u^2}$$

$u=1$ term.

$$= \left(\sum_{u=1}^{\infty} \frac{1}{u^2}\right) - \frac{1}{1^2}$$

Sometimes we can avoid negative terms in $\sum a_n$ by using $\sum |a_n|$ instead.

Thm If $\sum |a_n|$ converges, so does $\sum a_n$.

Pf: $a_n \leq |a_n| \ \forall n$ but we can't use comparison test. (why?)

Sketch of Pf: $s_n = a_1 + \cdots + a_n$

$t_n = |a_1| + \cdots + |a_n|$

$\sum |a_n|$ converges $\iff (t_n)$ converges $\iff t_n$ Cauchy

So we can make $t_n - t_m$ very small

$\forall \varepsilon > 0 \ \exists N \ni n > m > N \Rightarrow$

$$|t_n - t_m| < \varepsilon$$

$$\Big| \, |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \, \Big| < \varepsilon. \quad \circledast$$

We want to show $|s_n - s_m| < \varepsilon$ too. But

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq \circledast < \varepsilon.$$

Def • If $\sum |a_n|$ converges, then the series $\sum a_n$ converges absolutely.

• If $\sum a_n$ converges but $\sum |a_n|$ doesn't, then $\sum a_n$ converges conditionally

Ex $1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} + \frac{1}{128} - \frac{1}{256} + \cdots$

$$\left[\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = 1 + \tfrac{1}{2} + \tfrac{1}{4} \cdots = 2 \right]$$

$$\sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^n = 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots = \frac{1}{\underbrace{1-\left(-\frac{1}{2}\right)}_{3/2}} = \frac{2}{3}$$

For our series, $|a_n| = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$, $n \geq 0$. Since $\sum |a_n| = \sum \left(\frac{1}{2}\right)^n$ converges, our series conv's absolutely $\Rightarrow$ converges.

Ex $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$

converges but $\sum |a_n| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges (harm. series).

$(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ conv's cond'ly$)$

Thm 33.16 (Alternating Series Test)

If (a_n) decreasing and $a_n \to 0$,

then $\sum (-1)^{n+1} a_n$ converges!

Ex $\frac{1}{n} \to 0$ so $\sum (-1)^{n+1} \frac{1}{n}$ converges.

Ratio Test Let $a_n \neq 0$. $\circledast$

(a) if $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$ then Σa_n converges absolutely (hence conv's)

(b) if $\lim \left|\frac{a_{n+1}}{a_n}\right| > 1$ then Σa_n diverges

(c) Otherwise Ratio Test tells me nothing!!

Root Test

(a) if $\lim |a_n|^{1/n} < 1$ then Σa_n con's abs.

(b) if $\lim |a_n|^{1/n} >$ then Σa_n div's

(c) Otherwise, no information.

$\circledast$ $a_n \neq 0$, NOT $a_n \geq 0$.

Root, Ratio tests give same information.

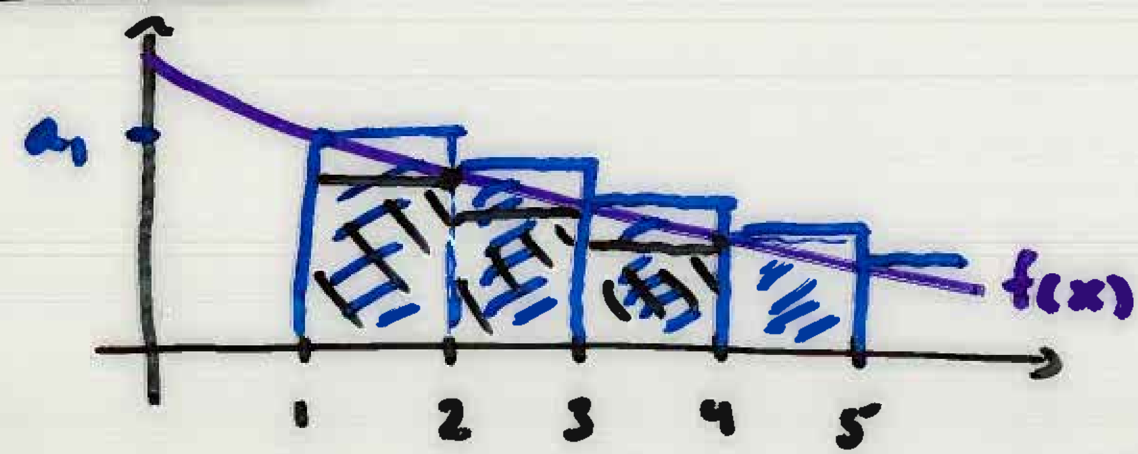
Exercise in §33: $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim |a_n|^{1/n}$

Integral Test Let $a_n = f(n)$, where $f:[0,\infty)\to\mathbb{R}$ positive, cont., decr'g.

Then $\sum a_n = \sum f(n)$ converges $\iff$

$\int_1^\infty f(x)\,dx = \lim_{a\to\infty}\int_1^a f(x)\,dx = L$, some real # L.

(Sketch) Pf:



Using left endpts $a_1+a_2+\dots+a_n \geq \int_1^n f(x)\,dx$.

Using right endpts, $a_2+a_3+\dots+a_n \leq \int_1^n f(x)\,dx$

if $\int_1^\infty f(x)\,dx$ exists (is finite), then $\sum a_n$ must converge as well.

if $\int_1^\infty f(x)\,dx = +\infty$, then $\sum a_n = +\infty$ too.

$\lim \left|\frac{a_{n+1}}{a_n}\right|$, $\lim |a_n|^{1/n}$, $\lim \left(\int_1^n f(x)\,dx\right)$

Determine convergence/divergence of:

1. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$ Ratio $\frac{(3/n+1)^{n+1}}{3^n/n^n} = \frac{3n^n}{(n+1)^{n+1}}$

Root: $\left|\left(\frac{3}{n}\right)^n\right|^{1/n} = \frac{3}{n} \to 0 < 1$

$\Rightarrow$ conv's abs'ly

2. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Ratio: $\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \to 0 < 1$

$\Rightarrow$ conv's abs'ly.

3. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\int_1^n \frac{1}{x^3}\,dx = \left.\frac{-x^{-2}}{2}\right|_1^n = -\frac{1}{2n^2} + \frac{1}{2}$

which converges as $n \to \infty$

$\Rightarrow \sum \frac{1}{n^3}$ conv's too.

4. $\sum_{n=1}^{\infty} \frac{n-1}{3n+2}$

$\to \frac{1}{3} \neq 0 \Rightarrow$ diverges!

Ex More generally $\sum \frac{1}{n^p}$ is called a p-series. Using Integral Test:

$$\int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx$$

$$= \left.\frac{x^{-p+1}}{-p+1}\right|_1^n$$

$$= \frac{n^{-p+1}}{1-p} - \frac{1}{1-p} = \frac{n^{1-p}}{1-p} - \frac{1}{1-p}$$

As $n \to \infty$ this quantity

is finite if $1-p<0 \iff p>1$.

infinite if $1-p>0 \iff p<1$.

By int. test, $\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p<1$.

($p=1$: $\sum \frac{1}{n}$ diverges: Harmonic series.)

Given $\sum a_n$, what test to use?

- If $a_n \not\to 0$ then $\sum a_n$ diverges.
- Known form: geometric series, p-series - use formulas.
- If sign oscillates ($\exists\ (-1)^n, (-1)^{n+1}$) Alt. Series Test.
- If it's "similar" to known series (geo, p, harmonic), try comp. test.

  Ex $\sum \frac{1}{3+2^n} \le \sum \frac{1}{2^n}$
- Factorials in formula? often ratio:

  Ex $\sum \frac{3^n}{n!}$ $\left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right| = \frac{3}{n+1} \to 0$
- If n appears as an exponent and especially in the base as well, try root test:

  $\sum \left(\frac{1}{n}\right)^n$ $\left|\frac{1}{n^n}\right|^{1/n} = \frac{1}{n} \to 0$

  $\sum \frac{1}{n^n}$

- If $a_n = f(n)$, f pos., decr'g, continuous **AND** $\int f(x)\,dx$ is nice, try Integral Test.

  Ex p-series.

⚠ Root and Ratio tests will always give the same conclusions b/c

$$\lim |a_n|^{1/n} = \lim \left|\frac{a_{n+1}}{a_n}\right|$$