§2.3 Functions

This section contains many def'n's, including "function" as a relation, i.e. subset of Cart. Product.

We'll cover this section in depth. You will make your future mathematical lives much easier if you put in the effort to learn:

1. $f: A \to B$ notation, domain, codomain, range
2. injective, surjective, bijective (onto, 1:1)
3. fn inverses, preimages
4. compositions.
Algebra Through Calculus

A function is a formula or rule which takes each input and transforms it to an output.

Inputs: $x, t, \theta$

Outputs: $f(x)$, $g(t) = y$

MV Calc/Lin. Alg./Etc

We use a more general notation.

$$f: A \rightarrow B$$

Inputs: Domain

Outputs: Target space

Codomain

Range ($f$) = set of actual outputs = $\{ f(x) \mid a \in A \}$

$\{ b \in B \mid \exists a \in A : f(a) = b \}$

$\Delta f$ must assign exactly one output to each element in domain.

$\Delta$ In many books, range = potential outputs, image = actual outputs.
Example: \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) i.e. \( f(x,y) = x^2 y^2 \)

- Domain = \( \mathbb{R}^2 \)
- Codomain = \( \mathbb{R} \)
- Range = \( \mathbb{R}^+ = \{ x \geq 0 \} \)

This book is even more general—at least, at first.

Definition: A function between sets \( A \) and \( B \) is a non-empty subset of \( A \times B \) (i.e. a relation) such that if \( (a,b) \) and \( (a,b') \) \( \in f \) then \( b = b' \).

Instead of giving formula or rule, this method lists all inputs with their corresponding outputs.

\( \circledast \) means each input has just one output.
Ex  $f: \mathbb{N} \rightarrow \mathbb{N}$, $f(n) = n+1$ becomes

$$f = \{(1,2), (2,3), (3,4), \ldots \}$$

$g: \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = \sin x$ becomes

$$g \neq \{(1, \sin 1), (0, \sin 0), (\frac{1}{2}, \sin \frac{1}{2}), \ldots \} \text{ (Can't "list out" } \mathbb{R})$$

$$g = \{(x, \sin x) \mid x \in \mathbb{R}\}$$

⚠️ In this defn of $f \subseteq A \times B$, don't need not be $A$!

Ex $A = \{2, 3, 4\}$ $B = \{\triangle, \square, \bigtriangleup\}$

$$f = \{(3, \triangle), (4, \square)\}$$ Here $2 \in A$, but $2 \notin \text{ dom } f$$

Usually we stick with $f: A \rightarrow B$, read "$f$ is a fn from $A$ to $B$". In this not'n, dom $f = A$. 
You're used to graphing fns:

We can also visualize them as "generic blobs."
Example \( f : X \to Y \)

\[ X = \text{set of 32830 students} \]
\[ Y = \{ \text{Rogness, Baker, Ewing, Honda, Ismail, Morawski} \} \]

\[ f(x) = \text{which instructor is hit by a tomato thrown by student } x. \]

\[ \text{range}(f) = \{ R, B, E \} \neq Y \]

\[ f(x) = \text{the person who writes exams} \]

Not surjective
Not injective
(178:1)
Example $f: X \to Y$

$X =$ set of 3283W students

$Y = \{\text{Rogness, Baker, Ewing, Honda, Ismail, Morawski}\}$

$f(x) =$ which instructor is hit by a tomato thrown by student $x$. 

$f(x)$ is a person grading your HW.... not surj, not injective
Example: $f: X \to Y$

$X =$ set of 32830 students
$Y =$ \{Rogness, Baker, Ewing, Honda, Ismail, Morawski\}

$f(x) =$ which instructor is hit by a tomato thrown by student $x$.

1st six students ensure everybody is hit; random after that.

Surjective, not injective
Hugely important properties of $f: A \to B$

**Def** $f$ is **surjective** (onto, is a surjection) if every potential output actually is an output, i.e. $\text{range}(f) = B$.

Relation Version: $\forall b \in B \exists a \in A \exists (a, b) \in f$.

Function Notation: $\forall b \in B \exists a \in A : f(a) = b$. 
Def \( f \) is injective (one to one, 1:1, an injection) if no two els are sent to same output.

Relation Version If \((a, b) \in f\) and \((a', b) \in f\) then \(a = a'\)

Function Notation If \(f(a) = b\) and \(f(a') = b\) then \(a = a'\) or if \(a \neq a'\) then \(f(a) \neq f(a')\) (i.e. the contrapositive)
Def: $f$ is a bijection (is bijective, is a 1:1 correspondence) if it is injective and surjective.

If $\exists$ bijection $A \rightarrow B$ it means $A, B$ are “equiv” (see §2.4) — essentially the same sets with elements relabeled.
"Redefining Fns"

There is no such thing as a fn which is not surjective — just poorly defined fns...

(My undergraduate advisor)

Ex. \( f: \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) = \cos x \).

Not surjective b/c
\[
\text{range}(f) = [-1, 1] \neq \mathbb{R}
\]

But if I redefine \( f: \mathbb{R} \rightarrow [-1, 1] \), \( f(x) = \cos x \)

f is now surjective without changing domain, any fn values, etc.

Not injective b/c \( \cos(0) = \cos(\pi) = 1 \). If I restrict domain to \( f: \left[ 0, \pi \right] \rightarrow [-1, 1] \), it's injective — but a "new" fn w/
Recall composition notation:

Thm 2.3.20 Let \( f: A \to B \) and \( g: B \to C \).

(a) \( f, g \) surjective \( \Rightarrow \) \( g \circ f \) of surjective

(b) \( f, g \) injective \( \Rightarrow \) \( g \circ f \) injective

(c)

⚠️ For the remainder of §2.3, a broken document camera forced the lecture onto the whiteboard. I’m still posting this “outline” as a reminder of what we covered.
Functions Acting on Sets

Def Let $f: A \to B$ and suppose $C \subseteq A$, $D \subseteq B$. Then
Example: \( f(x) = \cos x \)

\[ f^{-1}(1) = \]

\[ f^{-1}((0,1]) = \]

\[ f([\pi/2, 4\pi/3)) = \]
There are many theorems involving all of these concepts:

**Thm 2.3.16**

1. \( c \subseteq f^{-1}[f(c)] \)
2. \( f[f^{-1}(a)] \subseteq D \)
3. \( f(c_1 \cap c_2) \subseteq f(c_1) \cap f(c_2) \)

**Thm 2.3.18**

If \( f \) is injective, \( f(c_1 \cap c_2) = f(c_1) \cap f(c_2) \)
Def Let $f: A \rightarrow B$ be bijective. The inverse of $f$ is
Thm 2.3.24 Let $f: A \to B$ be bijective. Then

(a) $f^{-1}: B \to A$

(b) $f^{-1} \circ f = \text{id}_A$ and
Hotel Infinity

After years of working your way up through the ranks, you’ve finally achieved your goal: you’re the new manager of the Hotel Infinity! You are especially excited because the Hotel Infinity is one of a kind: it has infinitely many rooms, one for each whole number, arranged in an infinitely long hallway stretching off from the lobby:

On your first day, the owner asks if you have any questions.

“Just one,” you say. “When I was looking around, I noticed that there’s no way to change the VACANCY sign to NO VACANCY.”

“Ah yes,” says the owner, “I’m glad you noticed. That’s by design; if you run this hotel correctly, you’ll never need a NO VACANCY sign. If you ever turn a lodger away, I promise you: it will be your last day working at this hotel!”