Ex $\sim (0, 1)$

Idea mostly use $\text{id}: (0, 1) \to (0, 1)$ which sends $x$ to $x$ but need to send something to $0$. (Use $\frac{1}{2}$)

Creates a hole at $\frac{1}{4}$, so send $\frac{1}{4} \to \frac{1}{2}$, repeat.

Define $f: (0, 1) \to (0, 1)$, $f(x) = \begin{cases} x, & x \neq \frac{1}{2^n}, \text{same as } x \\ \frac{1}{2^{n-1}}, & x = \frac{1}{2^n}, n \in \mathbb{N} \end{cases}$
Thm 2.4.10 TFAE ("The following are equivalent")

(a) $S$ is cтble
(b) $\exists$ injection $f: S \rightarrow \mathbb{N}$.
(c) $\exists$ surjection $g: \mathbb{N} \rightarrow S$.

We commonly prove

(a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (c)

We prove (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) (3 pfs)

(b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (b) (4 pfs)

Sketch of (parts) of pf.

(a) $\Rightarrow$ (c).

$S$ cтble $\Rightarrow S = \{s_0, \ldots, s_n\}$ or $S = \{s, s_0, s_1, \ldots\}$

$S$ finite define $g: N \rightarrow S$, $g(1) = s_1, \ldots, g(n) = s_n$.

$S$ infinite $g(n) = s_n$
Putting it all together

If \( S \cap T \), they have the same \textbf{cardinality}.

So \( S \cup 3 \) and \( S(a,b,c) \) not equal as \textit{sets}, but same \textit{cardinality}.

The \textbf{cardinal number} or \textbf{cardinality} of a set is (informally) its size.

- \textit{cardinal number} of \( \emptyset \) is \( |\emptyset| = 0 \)
- \( \mathbb{N} = \{1, 2, 3, \ldots, n\} \) is \( |\mathbb{N}| = n \).
- \( |\mathbb{N}| = \aleph_0 \)
- \( |\mathbb{R}| = \mathfrak{c} \) ← \textit{continuum}
Def 2.4.14 Denote card’d number of S by |S|, so |S| = |T| iff S ∼ T, i.e. ∃ bijection f: S → T.

- Define |S| ≤ |T| if ∃ injection S → T.
- Define |S| < |T| if |S| ≤ |T| but not |S| = |T|.

I need to define b/c it’s not < with R; |S|, |T| could be ∞.

Working with these def’s not always as awful as it seems at first.
Thm 2.4.15 Let $S, T, U$ be sets.

(a) $S \subseteq T \Rightarrow |S| \leq |T|$

(b) $|S| \leq |S|$

(c) $|S| \leq |T|$ and $|T| \leq |U|$, then $|S| \leq |U|$

(d) $m, n \in \mathbb{N}$, $m \leq n$, then $|\text{Im}| \leq |\text{Im}|$

(c) $[\text{You read}]

m \leq n \Rightarrow \{1, 2, 3, \ldots, m\} \subseteq \{1, 2, m, m+1, \ldots, m\}$

use (a) again.
**Many Many Many**

**Def:** Given a set $S$, the power set of $S$, written $\mathcal{P}(S)$, is the set of all subsets of $S$.

**Ex:** $S = \{a, b, c, d\}$

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$$
Another way to count (not list them)

To construct a 4 subset of \{a,b,c,d\},
- For each elt, I need to decide whether it's included or not - Four Y/N questions

Total possibilities: \(2^4 = 16\)

Y/N is two options

\[\text{Thms. } |S| < |\mathcal{P}(S)|\]

\[\cdot |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| \quad (i.e. \quad 2^\aleph_0 = \mathfrak{c})\]

Corollary 3 infinite "chain" of larger and larger \(\aleph\)'s:

\[|\aleph| < |\mathcal{P}(\aleph)| < |\mathcal{P}(\mathcal{P}(\aleph))| < \ldots\]
One last curiosity

If $|S| \leq |T|$ and $|T| \leq |S|$, can we conclude that $|S| = |T|$?

Yes, but not by defn - by Schröder-Bernstein Thm.

exists $(0,1) \sim (0,1]$

$id_{(0,1)} : (0,1) \rightarrow (0,1]$ is inj'n $\Rightarrow |(0,1)| \leq |(0,1]|$

$f : (0,1] \rightarrow (0,1)$ not surj, is inj

$\Rightarrow |(0,1]| \leq |(0,1)|$

$\Rightarrow |(0,1)| = |(0,1]|$ by S.B. Thm.
A few words about §2.5, which covers Axioms of Set Theory

Math majors encouraged to read this section, but it's not officially part of this course.

Has the basic Axioms we use to build up set theory and modern mathematics.
Paradoxes: Some sets are members of other sets \( S \in B(S) \), even of themselves. Define

\[
B = \{ S : S \notin S \}
\]

Is \( B \) an el of itself? (i.e. is \( B \in B \))