2.4 Cardinality

Ok, the Hotel Infinity teaches us that infinity is weird. Especially when we compare sizes of infinite sets.

For finite sets it's easier! If $A = \{1,2,3\}$ and $B = \{0,\Delta,\Box\}$ then $A$ has 3 elts and $B$ has 3, so $B$ is "larger."
Ex. Write these sets in order from "smallest" (i.e. fewest members) to "largest" (most elts).

\[ N = \{1, 2, 3, \ldots\} \]
\[ N_0 = \{0, 1, 2, 3, \ldots\} \]
\[ \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \]
\[ Q^+ = \{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\} \]
\[ Q \]

all turn out to be same "size"

Irrationals: \( \mathbb{R} \setminus \mathbb{Q} \)
\[ \mathbb{R} \]
\[ \mathbb{C} \]
\[ (0, 1) \]
\[ [0, 1] \]

all turn out to be same "size"

But the sets in left col. have a different "size" than those on the right.
BIJECTIONS bring order to this chaos.

**Def** Two sets \( S, T \) are equinumerous if \( \exists \) bijection \( S \rightarrow T \). Write: \( S \sim T \)

Everything in \( T \) is “hit” by exactly one elt.

Idea if \( S \sim T \), they are in 1:1 correspondence and are the “same set” (with elts relabeled), hence the same size.
Ex: \{1, 2, 3\} and \{a, b, c\}

Define
\[
\begin{align*}
f(1) &= a \\
f(2) &= b \\
f(3) &= c
\end{align*}
\]

\{1, 2, 3\} ~ \{a, b, c\}

\{1, 2, 3, 4\} and \{a, b, c\}

No \textbf{bijection} possible, so these sets are different sizes.

\Rightarrow \text{ could choose outputs for } f(1), f(2), f(3), \text{ distinct but } f(4) \text{ will break injectivity.}

\{1, 2, 3, 4\} and \mathbb{N}

\textbf{No bijection possible}

\Rightarrow \text{ we can choose up to 4 different outputs, but will never be surjective.}
Ex \(N_0 = \{0, 1, 2, 3, \ldots\}\) \(\mathbb{N}\)

\[f: N_0 \to \mathbb{N}, \quad f(n) = n + 1.\]

Is \(f\) surj? Let \(m \in \mathbb{N}\). Then \(m = f(m-1) = (m-1)+1\).

Is \(f\) inj? Let \(x, y \in N_0, \ x \neq y\). Then \(x < y \Rightarrow x - 1 \neq y - 1\).

Ex \(\mathbb{N}, \mathbb{Z}\)

Want \(n, f(n)\) for \(f: \mathbb{N} \to \mathbb{Z}\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(f(n) = (-1)^n \lfloor \frac{1}{x} \rfloor - \text{floor} \quad \text{(round down)})</th>
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<td>1</td>
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<td>3</td>
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<td>4</td>
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<td>5</td>
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<td>7</td>
<td>-3</td>
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Ex. \( \mathbb{N}, \mathbb{Q}^+ = \{ \frac{p}{q} : q \geq 0 \} \)

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\( \frac{p}{q} \) is in \((p+1)st\) row \((p=0\text{ possible}) \)

\( f^{th} - \text{col} \)

Define \( f : \mathbb{N} \to \mathbb{Q}^+ \) as follows:

\( f(n) = n^{th} \text{ unique # we meet on this path.} \)

(We wrote \( f(1), f(2), \ldots, f(7) \) on board and discussed why this is a bijection.)
Def A set $S$ is.....

- finite if $S \sim \text{I}_n = \{1, 2, \ldots, n\}$
- denumerable if $S \sim \mathbb{N}$
- countable if $S$ is finite or denumerable.
- uncountable if $S$ is not countable.

In handy-dandy Venn Diagram in your book (p84)
Useful Consequence

Countable sets can be "listed in order"

- **Finite**: can write $S = \{s_1, s_2, s_3, \ldots, s_n\}$
- **Denumerable**: $S = \{s_1, s_2, s_3, s_4, s_5, \ldots\}$

$\exists f : \text{In} \to S$

$\text{In} \xrightarrow{f} S$

$f(1) = s_1$
$f(2) = s_2$
$f(3) = s_3$
$f(n) = s_n$

⚠️ No "canonical" order for which elt is $s_1$, which is $s_2$, etc.
Thm (Ex 2.4.11) Let $S, T$ be countable. Then $S \cup T$ is countable.

Proof: 3 cases: ① $S, T$ both finite.
② One is finite, one is denumerable
③ Both are denumerable.

Case 1: $S \sim \text{In}_n, T \sim \text{In}_k$

Define $f: \text{In}_n \rightarrow S \cup T$

$$f(p) = \begin{cases} h(p), & p \leq n \\ g(p-n), & p > n \end{cases}$$
Case 2 \text{ WLOG (without loss of generality) assume } S \text{ is finite, } T \text{ denumerable.}

\[ S \cup T = \{ s_1, s_2, \ldots, s_m, t_1, t_2, t_3, t_4, \ldots \} \]

\text{equinumerous with } \mathbb{N}.

Case 3 \text{ Suppose } f: \mathbb{N} \to S, \ g: \mathbb{N} \to T \text{ are bij'ns.}

\text{define } h: \mathbb{N} \to S \cup T \text{ by}

\[ h(n) = \begin{cases} f\left\lceil \frac{n+1}{2} \right\rceil, & n \text{ odd} \\ g\left( \frac{n}{2} \right), & n \text{ even} \end{cases} \]

\text{Left for you in each case: what if } S \cup T \neq \emptyset ?
Thm (Practice 2.4.2) \( \sim \) is an equiv. reln.

Refl id\(_x\)\( : X \rightarrow X \) bij.  Sym: \( f : X \rightarrow Y \) bij \( f^{-1} : Y \rightarrow X \) bij.

Trans: composition of bij\'s is bij

Corollary \( \mathbb{N} \sim \mathbb{Q} \)

\( \mathbb{P} : \mathbb{N} \sim \mathbb{Q}^+ \sim (\mathbb{Q}^+ \cup \mathbb{Q}) \sim (\mathbb{Q}^+ \cup \mathbb{Q}) \cup \mathbb{Q}^- \)

\( (\mathbb{N} \sim \mathbb{Q}^-) \) by same "path" argument as \( \mathbb{Q}^+ \ldots \)

Thm 2.4.3 Any subset of a countable set \( S \) is countable
Thm 2.4.13 IR is uncountable

(Part of your intellectual heritage!)

Pf: Using CP of previous thm, we’ll show \((0,1)\) is uncountable \(\Rightarrow IR\) is uncountable.

Assume \((0,1)\) is countable, so we can list its elts. in order:

\[ x_1 = 0. \, x_{12} \, x_{13} \, x_{14} \, x_{15} \, x_{16} \cdots \]

\[ x_2 = 0. \, x_{21} \, x_{22} \, x_{23} \, x_{24} \, x_{25} \cdots \]

\[ x_3 = 0. \, x_{31} \, x_{32} \, x_{33} \, x_{34} \cdots \]

\[ x_4 = 0. \, x_{41} \, x_{42} \, x_{43} \, x_{44} \cdots \]

Define \( b = 0. \, b_1 \, b_2 b_3 b_4 b_5 \cdots \) by \( b_n = \begin{cases} 2, & x_{nn} \neq 2 \\ 3, & x_{nn} = 2 \end{cases} \)

\( \Rightarrow b \) is not in my list by construction, even though \( b \in (0,1) \).