§3.3 The Completeness Axiom

Another section with a lot of topics:

- supremum, infimum, bounds
- Completeness Axiom
- Archimedean Property of \( \mathbb{R} \)
- Density

and... "style" of proofs w/ sup, inf. (w/ \( \epsilon \)...)
**Bounds**

**Ex** what are the max, min elts of....

\[ S = \{0, 2, 4\} \quad \text{min} = 0 \quad \text{max} = 4 \]

\[ S = [0, \infty) \quad \text{min} = 0 \quad \text{no max}. \]

\[ S = (0, 1) \quad \text{no min} \quad \text{no max} \]

\[ S = \{1 - \frac{1}{k} \mid n \in \mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \quad \text{min} = 0 \quad \text{no max} \]

**Def** \( \text{mes } S \) is \( \text{min } S \) if \( m \leq s \) \( \forall s \in S \).

\( \text{mes } S \) is \( \text{max } S \) if \( s \leq M \) \( \forall s \in S \).
More generally....

- $m$ lower bound for $S$ if $m \leq s \forall s \in S$
- $M$ upper bound for $S$ if $s \leq M \forall s \in S$

**Example**

- $S = \{0, 2, 4\}$  
  - 0 is min and lower bd.  
  - 4 is max, hence U.B.
  - $-10$, $-\pi$, $-10^{26}$ also
  - Other lds include 5, 10, 39, ... lower lds.

- $S = [0, 1)$  
  - No max, but 1 is upper bound

- $S = \{1 - \frac{1}{n} | n \in \mathbb{N}\}$  
  - (so is 9, 10, 39, ...
Observations

1. $m$ is an upper bound for $S$ $\Rightarrow$
   so is any $\#$ larger than $m$.

2. Similar for lower bds: if $m$ is L.B.
   for $S$, so is any $\# < m$.

3. min/max automatically a L.B/U.B
   Conversely, a L.B/U.B in the set is
   automatically a min/max.

Q: can we find the "best" L.B/U.B for $S$?
Def Let $\emptyset \neq S \subseteq \mathbb{R}$. If $S$ is bounded above, then the **supremum** of $S$ is its **least upper bound**, denoted: $\text{sup } S = \text{ lub } S$.

Hence $m = \text{sup } S$ iff:

(a) $m$ is upper bound: $m \geq S \forall x \in S$

(b) anything smaller than $m$ is not an upper bd:

If $m' < m$ it is not an upper bd, i.e. if $m' < m$ then $\exists s' \in S$ s.t. $m' < s$

**Ex** $S = (0,1)$. Claim that $1 = \text{sup } S$.

(a) $S = \{0 < x < 1\} \Rightarrow 1$ is upper bound.

(b) any # less than 1 is not least u.b. If $u \in (0,1)$, then $\exists u \epsilon (0,1)$ is bigger than u.
Similarly, if $S \in \mathbb{R}$ is non-empty and bounded below, its greatest lower bound or infimum $(\inf S = \text{g.l.b.})$ satisfies:

(a) it is a lower bound
(b) no larger # is a lower bound.

**Ex**

$S = [0,1]$

$\inf S = 0$
$\sup S = 1$

$S = (0,\infty)$

$\inf S = 0$

Sup $S$ does not exist.
(S has no upper bound, hence no least upper bound.)

$S = \left\{ \frac{1}{n} | n \in \mathbb{N} \right\} = \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}$

$\inf S = 0$
$\sup S = 1.$

You can often find $\inf S$, $\sup S$ intuitively or "by inspection." Proving your answers are correct is trickier, especially part (b) of the def.

**Ex** to prove $m = \sup S$, must

1st show $m$ is an upper bd of $S$ - usually with algebra.

Then must show any smaller $m'$ (i.e. $m' < m$) is not an upper bd.

**OFTEN** we choose small $\varepsilon > 0$, set $m' = m - \varepsilon$.
Ex (Spring 2010 Exam Prob) \( A = \left\{ \frac{n-1}{n+1} \mid n \in \mathbb{N} \right\} \)

So \( A = \left\{ \frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots \right\} = \left\{ 0, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots \right\} \)

(i) Show \( A \) bdd below, above.

\( n \in \mathbb{N} \Rightarrow n-1, n+1 \ both \geq 0 \Rightarrow \frac{n-1}{n+1} \geq 0. \) Hence \( A \) bdd below by 0.

Also, \( \forall n, n-1 < n+1 \Rightarrow \frac{n-1}{n+1} < 1. \) Hence \( A \) bdd above by 1.

(ii) Find \( l = \inf A, m = \sup A. \) (Prove your answers!)

\( l = \inf A = 0, \ m = \sup A = 1. \)

Proof that \( \inf A = 0 \). First, as shown in (i), \( l = 0 \) is a lower bound of \( A \), because \( 0 \leq a \ \forall a \in A. \) But it is also the greatest lower bound, because any larger number will not be a lower bound: If \( l' > 0 \), then it's not a lower bd because 0 itself is in \( A \) and less than \( l' \).
Proof that $m = \sup A = 1$: From (i) we know $m = 1$ is upper bound.

To prove it's $\sup A$, we must show any # less than $m = 1$ cannot be an upper bound.

Take (presumably small) $\varepsilon > 0$, so $1 - \varepsilon < 1$, and show

$\exists \ n \in \mathbb{N}$ such that $1 - \varepsilon < \frac{n-1}{n+1} < 1$.

clt of $A$,
so $1 - \varepsilon$ not an upper bound.

If that's true for any $\varepsilon > 0$, then we have shown no # less than 1 can be upper bound, hence $m = 1$ is $\sup A$.

These types of proofs usually have a "Think" portion and a "proof" portion.
Repeat to show sup A=1, show: \( A > 0 \in \mathbb{R} \text{ s.t. } 1 - \varepsilon < \frac{n-1}{n+1} \leq 1 \)

which implies \( 1 - \varepsilon \) not upper bound.

Use algebra to "solve" for \( n \) in terms of \( \varepsilon \):

\[
1 - \varepsilon < \frac{n-1}{n+1} = \frac{(n-1)+2}{n+1} = \frac{(n+1)-2}{(n+1)} = 1 - \frac{2}{n+1}
\]

\[
-\varepsilon < -\frac{2}{n+1}
\]

\[
\varepsilon > \frac{2}{n+1}
\]

\[
n+1 > \frac{2}{\varepsilon}
\]

\[
n > \frac{2}{\varepsilon} - 1
\]

"Think" portion of this problem:

So any \( \frac{n-1}{n+1} \) (with \( n > \frac{2}{\varepsilon} - 1 \)) will be larger than \( 1 - \varepsilon \),

so \( 1 - \varepsilon \) not an upper bound.
For actual proof, use what we found in algebra and write it in reverse!

Proof that \( \sup \left\{ \frac{n-1}{n+1} : n \in \mathbb{N} \right\} = 1 \): \[
= A
\]

**Proof**

First note that, \( \forall n \in \mathbb{N}, n-1 < n+1 \). Hence \( \frac{n-1}{n+1} < 1 \implies 1 \) is an upper bound of \( A \).

To show 1 is the supremum of \( A \), we must also show that \( \forall \varepsilon > 0, 1 - \varepsilon \) is not an upper bound of \( A \).

Given any \( \varepsilon > 0 \), choose \( n > \frac{2}{\varepsilon} - 1 \), which is possible by the Arch. Prop.

Then \( n+1 > \frac{2}{\varepsilon} \)

\[ 1 > \frac{2}{n+1} \]

\[ \varepsilon > \frac{2}{n+1} \]

\[ 1 - \varepsilon < 1 - \frac{2}{n+1} = \cdots = \frac{n-1}{n+1} \cdot \varepsilon \cdot A. \implies 1 - \varepsilon \text{ is not an upper bound of } A. \]
IR is "complete", which means it satisfies:

**Completeness Axiom** Every non-empty subset of IR which is bounded above has a supremum.

Notes

1. IR is not complete.

\[ S = \{ q \in IR \mid q^2 < 2 \} \subseteq IR \] is non-empty (Check: 0 \in S) and bounded above (Check: 2 is an upper bound. So is 10, or 3.141...) but sup S would be \( \sqrt{2} \notin IR \).

2. Also implies every non-empty subset of IR which is bounded below has an infimum. (Can you see why? Hint: def T = \{ -s \mid s \in S \} to convert "bddd below" to "bddd above."
Thm 3.3.9 (Archimedean Property of IR)

N is unbounded in IR.

(Pf in book)
Thm 3.3.10  TFAE

(*) Archimedean Property

†  (a) \( \forall z \in \mathbb{R} \ \exists n \in \mathbb{N} \ \text{s.t. } n > z \)

† (b) \( \forall x > 0 \ \forall y \in \mathbb{R} \ \exists n \ \text{s.t. } nx > y \)

† (c) \( \forall x > 0 \ \exists n \in \mathbb{N} \ \text{s.t. } 0 < \frac{1}{n} < x \).

To save time, prove \((*) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (*)\)

(b) \Rightarrow (a) prove by contradiction. If no such \( n \) exists, then \( z \) would be an upper bd of \( \mathbb{N} y \).

(a) \Rightarrow (b) Choose \( z = y/x \) and apply (a). \( n > z, \ n > y/x, \text{ and} \)

(b) \Rightarrow (c) Let \( y = 1 \) in (b): \( nx > 1, \text{ so } x > \frac{1}{n} \)

(c) \Rightarrow (*) (You think about)
Finally... 

Thm 3.3.13 \( \mathbb{Q} \) is dense in \( \mathbb{R} \): \( \forall x, y \in \mathbb{R}, \ x < y, \) there exist \( r \in \mathbb{Q} \) s.t. \( x < r < y \): \( x \) \( \longrightarrow \) \( y \)

Proof: Read book.

Constructive Method: Start writing out decimal expansions: 

\[ x = 1.414234567 \ldots \]
\[ y = 1.414256789 \ldots \]

Find 1st place they differ, split difference:

\[ r = 1.4143 = \frac{14143}{10000} \]
Also...

Thm 3.3.15 \( \forall x, y \in \mathbb{R}, x < y \exists \text{ irrational } z \in \mathbb{R} \setminus \mathbb{Q} \) such that \( x < z < y \).

(irrational numbers are dense in \( \mathbb{R} \)).

**Pf:** By density of \( \mathbb{Q} \), \( \exists r \in \mathbb{Q} \) s.t.

\[
\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}
\]

\( \Rightarrow x < r \cdot \sqrt{2} < y \)

(rational number \( r \) is always irrational?)