§ 3.4 Topology of IR

In this context, “topology” refers to open and closed sets in $\mathbb{R}$.

With sequences, limits we talk about “points close to $x$”. This section lays groundwork for putting “close to” on a rigorous footing:

* neighborhood
* interior pts, boundary pts
* open, closed sets in $\mathbb{R}$
* accumulation pts, closure
Def Let \( x \in \mathbb{R}, \varepsilon > 0 \). Then

\[
N(x; \varepsilon) = \{ y \in \mathbb{R} \mid |y - x| < \varepsilon \} = \{ y \in \mathbb{R} \mid |x - y| < \varepsilon \}
\]

is the neighborhood (nbhd) centered at \( x \) with radius \( \varepsilon \). Also \( \exists \) "punctured" nbhd:

\[
N^*(x; \varepsilon) = \{ y \in \mathbb{R} \mid 0 < |y - x| < \varepsilon \}
\]

Equivalently

\[
N(x; \varepsilon) = (x - \varepsilon, x + \varepsilon)
\]

\[
N^*(x; \varepsilon) = (x - 3, x) \cup (x, x + 3)
\]
\[ N(x; 3) = \{ 1 \text{y} - x < 3 \} \quad N^*(x; 3) = \{ 0 < 1 \text{y} - x < 3 \} \]

\[ \begin{align*}
\text{Ex} \quad N(2; 1) &= (1, 3) \\
N(0; \frac{1}{2}) &= (-\frac{1}{2}, \frac{1}{2}) \\
N^*(10; 3) &= (7, 10) \cup (10, 13)
\end{align*} \]

**Def** \( x \in \mathbb{R} \) is an *interior point* of \( S \) if \( \exists \varepsilon > 0 \) such that \( N(x; \varepsilon) \subseteq S \).

\[ N(1; 4) = (-3, 5) \notin S, \text{ but } N(1; \frac{1}{2}) \in S, \text{ so} \]

\[ \text{Ex} \quad (0, 3) \]

1 is int. pt. of \( S \)

3 not int pt. Not in \( S = (0, 3) \), and any nbhd of 3 includes pts not in \( S \).
Conversely, if every nbhd of $x$ contains pts in $S$ ($N_n S \neq \emptyset$) and also contains pts not in $S$ ($N_n S^c \neq \emptyset$) then $x$ is a boundary point of $S$.

**Ex 3** is bdg pt of $(0,3)$.

Every $N(3; \varepsilon) = (3-\varepsilon, 3+\varepsilon)$ includes pts in $(0,3)$ and not in $(0,3)$.

**Def** \( \text{int } S = \text{ set of all interior pts of } S \)

\( \text{bd } S = \text{ set of all bdg pts of } S \).
Ex: \( S = \{0, 2, 4\} \)

No int pts, \( \text{int} S = \emptyset \)

\( 0 \in \text{bd} \ S, \text{be any N}(0; \epsilon) \) contains pts not in \( S \) (neg #s) and pt \((0)\) in \( S \). 2, 4 \( \in \text{bd} \ S \).

\[ T = [0, 1) \]

\( \text{int} \ T = (0, 1) \)

\( \text{bd} \ T = \{0, 1\} \)

\( \text{int} \) pts must be in the set; bd pts need not be!
Open/Closed Sets

⚠️ Our approach slightly different than the book's; everything will work out the same, but our def's are the books thm's and vice-versa.

Def \( S \subseteq \mathbb{R} \) is open in \( \mathbb{R} \) if every \( x \in S \) is an interior point.

- equivalently, \( S \subseteq \text{int } S \).
- Since \( \text{int } S \subseteq S \), could also say \( S = \text{int } S \).
Examples

1. Any interval \((a, b) = \{ a < x < b \}\) is open.

   Let \( \varepsilon = \min \{ d_1, d_2 \} \)
   
   \[
   = \min \{ |x-a|, |x-b| \} \Rightarrow N(x; \varepsilon) \subseteq (a, b).
   \]

2. Since \( N(x; \varepsilon) = (x-\varepsilon, x+\varepsilon) \) (an open interval)
   
   by part 1, \( N(x; \varepsilon) \) is open
   "open nbhds."

3. \( \mathbb{R} \)
   Let \( x \in \mathbb{R} \). Any nbhd \( N(x; \varepsilon) \subseteq \mathbb{R} \)
   so \( \mathbb{R} \) is open.
4) \( \emptyset \subseteq \mathbb{R} \) open for "trivial" reasons:

if \( x \in \emptyset \) then \( x \) is int pt of \( \emptyset \).

always \( F \), hence implication is \( T \).

5) \( S = (0,1) \cup (3,4) \)

Let \( x \in S \), so \( x \in (0,1) \)

or \( x \in (3,4) \)

If \( x \in (0,1) \), which is an open set, then \( \exists N(x; \varepsilon) \subseteq (0,1) \subseteq S \).

If \( x \in (3,4) \), then because \( (3,4) \) is open, \( \exists \varepsilon > 0 \)

s.t. \( N(x; \varepsilon) \subseteq (3,4) \subseteq S \).
\[ S = (0,5) \cap (2,6) \]

\[ = (2,5), \text{ open} \]

Let \( x \in S = (0,5) \cap (2,6) \). Then \( x \in \mathbb{R} \) and \( x \in (2,6) \).

Because \( x \in (0,5) \), \( \exists \varepsilon_1 > 0 \) s.t. \( N(x; \varepsilon_1) \subseteq (0,5) \).

Because \( x \in (2,6) \), \( \exists \varepsilon_2 > 0 \) s.t. \( N(x; \varepsilon_2) \subseteq (2,6) \).

Key if \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2 \} \), then \( N(x; \varepsilon) \subseteq S \). Hence \( S \) is open.

\[ ! S = (0,1) \cap (3,4) = \emptyset \text{ which is still open.} \]
More generally...

**Thm 3.4.10**

(a) Any union of open sets is open. \( \implies \) finitely or infinitely many (!!!)

(b) An intersection of finitely many open sets is open.

Ex: \( \bigcup_{n \in \mathbb{N}} (0, n) = (0, 1) \cup (0, 2) \cup (0, 3) \cup \cdots = (0, \infty) \) open

\( \bigcap_{n \in \mathbb{N}} (-\frac{1}{n}, 1 + \frac{1}{n}) = (-1, 2) \cap (-\frac{1}{2}, \frac{3}{2}) \cap (-\frac{1}{3}, \frac{4}{3}) \cdots = [0, 1] \) NOT open
Def: $S \subseteq \mathbb{R}$ is closed if $\mathbb{R} \setminus S = S^c$ is open.

⚠️ closed, open NOT opposites.

$[0,1)$ is neither:

not open because 0 not int. point.

not closed b/c $[0,1)^c = (-\infty,0) \cup [1,\infty)$

not open

(1 is not int. pt)
A Topologist "Opens" a Store....
Examples

1. \( S = [0,1] \) closed b/c \( [0,1]^c = \mathbb{R} \setminus [0,1] \)
   \[ S^c \text{ open } \Rightarrow S \text{ closed.} \]

2. \( S = \emptyset \).
   \[ S^c = \emptyset^c = \mathbb{R} - \emptyset = \mathbb{R} \],
   which is open - hence \( \emptyset \) is closed.

3. \( \mathbb{R} \).
   \[ \mathbb{R}^c = \mathbb{R} - \mathbb{R} = \emptyset \text{ open, which means} \]
   \( \mathbb{R} \text{ is closed.} \)

\( \emptyset, \mathbb{R} \) both open, closed: they are clopen.
a different characterization.

Thus \( S \subseteq \mathbb{R} \) is closed iff it contains all its boundary points (\( \text{bd} \, S \subseteq S \)).

Example: \([a, b]\) is closed because \([a, b] = (-\infty, a) \cup (b, \infty)\) is open.

\( \text{bd} \, [a, b] = \{a, b\} \subseteq [a, b] \)

0, 2, 43. \( \text{bd} \, \{0, 2, 43\} = \{0, 2, 43\} \subseteq \{0, 2, 43\} \)
Thm (a) Any (finite or infinite) intersection of closed sets is closed.

(b) Any finite union of closed sets is closed.

Pf (b) Suppose $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}$ are closed.

$\bigcup_{k=1}^{\infty} A_k$ is closed if its comp. is open.

$\left( \bigcup_{k=1}^{\infty} A_k \right)^c = \bigcap_{k=1}^{\infty} (A_k^c)$ is open b/c its finite $\cap$ of open sets.
How about $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}$

Claim: every $a \in A$ is a body pt.
Any nbhd $N_a$ ct'd at $\frac{1}{n}$ will include pts not in $A$.

Claim: $0$ also a body pt.
Any nbhd $N(0; \varepsilon)$ includes pts not in $A$ (like neg. #'s) but also includes pts in $A$.

$\forall \varepsilon > 0, \exists n \geq 0 < \frac{1}{n} < \varepsilon$, so $\frac{1}{n} \in N(0; 3)$ (doesn't include all its body pts)

$\Rightarrow A$ not closed.
Int. U's of closed sets can fail to be closed.

\[ \bigcup [-n, n] = \cdots = \mathbb{R} \text{ open but still closed.} \]

\[ \bigcup \left[ \frac{1}{n}, 2 - \frac{1}{n} \right] = \left[ 1, 1 \right] \cup \left[ \frac{1}{3}, \frac{3}{2} \right] \cup \left[ \frac{1}{5}, \frac{5}{3} \right] \]

\[ \cap \mathbb{N} = (0, 2) \]

\[ \bigcup \left[ \left[ 0, 1 \right] \cap \mathbb{N} \right] \]
Not in course, but read if you're interested

3 notion of "closure" of a set.
Also a hybrid of int/bd pts called "accumulation" pts.

Also, §3.5 Compact Sets not in course but....

Def A set \( \mathbb{S} \subseteq \mathbb{R} \) is compact if it is closed and bounded (above and below).

(Not actually def\(^1\) - this is Heine-Borel Thm).