§ 8.2 Convergence Tests

**Goal:** Can we analyze $a_n$'s instead of $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ to determine if $\sum a_n$ converges?

Theorem names in this section might sound familiar... comparison test, ratio test, root test, integral test, alternating series test.
Basic Test for divergence comes from

Then $\sum a_n$ converges $\Rightarrow a_n \to 0$

C.P. $a_n \to 0 \Rightarrow \sum a_n$ diverges.

Example:

$$\sum_{n=1}^{\infty} \frac{(n+1)!}{(n-1)!} = \frac{2!}{0!} + \frac{3!}{1!} + \frac{4!}{2!} + \frac{5!}{3!} + \cdots$$

$$a_n = \frac{(n+1)!}{(n-1)!} = \frac{(n+1)n(n-1)(n-2)\cdots}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{n(n+1)}{3\cdot 2\cdot 1} \to 0$$

Always worth checking by it's quick but most series $\sum a_n$ we'll see will have $a_n \to 0$. 
Thm 8.2.1 (Comparison Test). $0 \leq a_n \leq b_n \ \forall n.$

(a) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

(b) $\sum a_n = +\infty$ $\Rightarrow$ $\sum b_n = +\infty$

⚠️ book switches roles of $a_n, b_n$ in (a)

Pf: Let $S_n = a_1 + a_2 + \cdots + a_n$
$t_n = b_1 + b_2 + \cdots + b_n$
$\Rightarrow S_n \leq t_n$ since $a_k \leq b_k \ \forall k.$

(a) $a_n \geq 0 \Rightarrow S_n$ increasing.

By earlier sections, $S_n \leq t_n \ \forall n \Rightarrow \lim S_n \leq \lim t_n = t.$

$\Rightarrow S_n$ bounded above by $t,$ below $t_n \rightarrow t.$

by $S_t$ (since it's increasing) $\Rightarrow$ by MCT
$S_n$ converges $\Rightarrow$ $\sum a_n$ converges.
Key to using Comparison Test
(a) Compare to a known series.
(b) Make sure direction of comparison is useful.

Example: \[ 0 \leq \frac{1}{n^2} \leq \frac{1}{n} \quad \forall n \quad \text{so} \]
\[ 0 \leq \sum \frac{1}{n} \leq \sum \frac{1}{n} = +\infty \]
so comp test tells us nothing about \( \sum \frac{1}{n^2} \) in this case.

Example: \[ \sum \frac{1}{n(n+1)} = 1 \quad \text{(telescoping)} \]
\[ \forall n \quad 0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2} \]
\[ \Rightarrow \quad 0 \leq \sum \frac{1}{n(n+1)} \leq 3 \cdot \frac{1}{n^2} \leq 0 \leq \frac{1}{n^2} \quad \text{not a useful comparison.} \]
\[ \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2} \quad \forall n \]

and \( \sum \frac{1}{n(n+1)} \) converges to 1

\( \Rightarrow \sum \frac{1}{n^2} \) converges by comp. thm. (test).

To what?

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots \right) - 1 \]

\[ = \left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots \right) - 1 \]

\[ = \frac{\pi^2}{6} - 1 \quad \text{(no proof)} \]

All.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=2}^{\infty} \frac{1}{u^2} \quad \text{u} = n+1 \]

eq \cdots
Thm: If $\sum |a_n|$ converges, so does $\sum a_n$.

Notes:
1. If $\sum |a_n|$ converges, we say the series $\sum a_n$ converges absolutely.
2. If $\sum a_n$ converges but $\sum |a_n|$ doesn't, we say $\sum a_n$ converges conditionally.

Example:
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$ but $3 \left| \frac{(-1)^n}{n} \right| = \frac{3}{n} \rightarrow +\infty$.

Alternatively, harmonic series converges conditionally, but not absolutely.
Pf that $\Sigma a_n$ converges $\Rightarrow$ $\Sigma a_n$ converges

Sketch of Pf: $S_n = a_1 + a_2 + \ldots + a_n$
$T_n = |a_1| + |a_2| + \ldots + |a_n|$

$a_n \in \{a_n\}$ and $S_n \leq T_n \ \forall \ n$, but comparison tests
are useless.

$\Sigma a_n$ converges $\Leftrightarrow$ $T_n$ converges $\Leftrightarrow$ $T_n$ Cauchy.
So we can make $T_n - T_m$ small.

Given $\varepsilon > 0$, $\exists \ N \ s.t. \ n > m > N \Rightarrow$

$|T_n - T_m| = |(a_{m+1} + \ldots + a_n)| < \varepsilon.$

$|S_n - S_m| = |(a_1 + \ldots + a_m) - (a_1 + \ldots + a_n)| < \varepsilon.$

Thus $|S_n - S_m| < \varepsilon.$
Other Tests

Thm (Alternating Series Test) If \( a_n \to 0 \) and is decreasing then \( \sum (-1)^n a_n \) converges.

[Ex: \( \frac{1}{n} \) decr's, \( \frac{1}{n} \to 0 \) so All. Harm. Series conv's]

Thm Ratio Test

(a) if \( \lim \left| \frac{a_{n+1}}{a_n} \right| < 1 \) then \( \sum a_n \) converges absolutely.

(b) if \( \lim \left| \frac{a_{n+1}}{a_n} \right| > 1 \) then \( \sum a_n \) diverges.

(c) Otherwise, no information.

\[ \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| a_n \right|^{\frac{1}{n}} \]

Thm Root Test

(a) if \( \lim \left| a_n \right|^{\frac{1}{n}} < 1 \) then \( \sum a_n \) converges abs.

(b) if \( \lim \left| a_n \right|^{\frac{1}{n}} > 1 \) then \( \sum a_n \) diverges

(c) else no info!
**Remember**: The Ratio and Root Tests in Section 8.2 use something called “lim inf” and “lim sup.”

We didn’t cover Section 4.4 on Subsequences, which is where “lim inf” and “lim sup” are introduced.

Thus I gave you variants of the Ratio and Root Tests which use regular limits; those variants are what we will use on the Final Exam.
**Integral Test** Let $a_n = f(n)$, where $f: [0, \infty) \to \mathbb{R}$ is positive, continuous and decreasing. Then

$$\sum a_n = \sum \Xi f(n) \text{ converges } \iff \int_1^\infty f(x) \, dx \text{ converges}$$

Recall

$$\int_1^\infty f(x) \, dx = \lim_{b \to \infty} \int_1^b f(x) \, dx$$

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(Sketch of) $f$

Using **left** endpts,

$$a_1 + a_2 + a_3 + \cdots + a_n \geq \int_1^n f(x) \, dx$$

if $\int_1^\infty f(x) \, dx = +\infty$, then $\sum a_n = +\infty$ too.

Using **right** endpts,

$$a_2 + a_3 + \cdots + a_n \leq \int_2^n f(x) \, dx.$$

If $\int_1^\infty f(x) \, dx$ is finite, then $\sum a_n$ converges too.
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|, \lim_{n \to \infty} \left| a_n \right|^{1/n}, \lim_{x \to \infty} \int_{0}^{x} f(x) \, dx \]

\[ \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \]

Ratio \[ \left| \frac{(\frac{2}{3})^{n+1}}{(\frac{2}{3})^n} \right| = \frac{3^{n+1}}{3^n} \cdot \frac{n}{3^n} = \frac{3^n}{(3/2)^n} \]

Root \[ \left| (\frac{2}{3})^n \right|^{\frac{1}{n}} = \frac{2}{3} \rightarrow 0 < 1 \Rightarrow \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \text{ converges absolutely.} \]

\[ \sum_{n=1}^{\infty} \frac{3^n}{n!} \]

Ratio \[ \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \rightarrow 0 < 1 \Rightarrow \text{ conv. absolutely.} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} \]

\[ \int_{1}^{\infty} \frac{1}{x^3} \, dx = \frac{1}{x^2} \bigg|_{1}^{\infty} = -\frac{1}{2a^2} + \frac{1}{2} \rightarrow \frac{1}{2} \text{ as } a \to \infty \]

\[ \sum_{n=1}^{\infty} \frac{n-1}{3n^2} \rightarrow \frac{1}{3} \neq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{n-1}{3n^2} \text{ diverges.} \]

\[ \sum_{n=1}^{\infty} \frac{n^2+2}{3n^3} \]

\[ \frac{n^2+2}{3n^3} = \frac{\frac{1}{n} + \frac{2}{n^3}}{3} \text{ Check with A.S.T.} \]
Ex More generally, $\sum \frac{1}{n^p}$ is called a $p$-series.

Using integral test:

$$\int_1^n \frac{1}{x^p} \, dx = \int_1^n x^{-p} \, dx$$

$$= \left. \frac{x^{-p+1}}{-p+1} \right|_1^n = \frac{n^{1-p}}{1-p} - \frac{1}{1-p}$$

- This converges as $n \to \infty$

As $n \to \infty$ this quantity is finite if $1-p>0$ ($p>1$)

Infinite if $1-p<0$ ($p<1$)

By int test $\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p<1$

($p=1$: Harmonic Series, diverges)
Given $\sum a_n$, which test to use?

- If $a_n \rightarrow 0$ then $\sum a_n$ diverges.
- Known form: geometric series, $p$-series — use formulas.
- Check if it's telescoping series
- If $a_n$ has $(-1)^n$ or $(-1)^{n+1}$ (etc), Alt. Series Test.
- If it's "similar" to known series (geo, p, harm)
  try comparison test:

  $\sum \frac{1}{3 + 2^n} \leq \sum \frac{1}{2^n}$
• If $n$ appears as exp (and especially in base too), root test:

$$
\leq \left(\frac{1}{a}\right)^n 
$$

etc.

• Factorials, exp formulas: Ratio Test.