§1.2 Quantifiers

Quantifiers... quantify things!

In math, we're usually interested in whether something is true:

- always $x^2 \geq 0$
- sometimes $x^2 > 0$ (at least once)
- never $x^2 < 0$

Since we're lazy, we use symbols...
Existential Quantifiers

\( \exists \): there exists (at least one)

\( \exists ! \): there exists a unique (exactly one) (not universal)

\( \forall \): there does not exist (slang)

Universal Quantifier

\( \forall \): for all, for every.

Other Notation

\( \exists \): such that (sometimes \( :) \), \( ! \), esp. with sets

\( p(x) \): stmt whose truth value depends on value of \( x \).

\( \exists \: p(x) : x^2 - 1 = 0 \quad p(1) : \text{true} \quad p(2) : \text{false} \)
Ex Write these stmts with symbols

① For some x, \( x^2 - 1 = 0 \). \( \exists x \in \mathbb{R} \exists x^2 - 1 \)

② For every real number \( x > 0 \), there is a real number \( y \) such that \( y^2 = x \).

\[ \forall x > 0, \exists y \in \mathbb{R} \exists y^2 = x. \text{ or } \forall x > 0, \exists y \in \mathbb{R} \exists y^2 = x \text{ or } \]

③ Every real number has a cube root.

\[ \forall x, \exists y \exists y^3 = x. \text{ or } \forall x \exists y \exists y^{1/3} \]

④ Given any number, there is a larger number.

\[ \forall x \exists y \exists y > x \text{ (y can depend on x)} \]
There exists a largest #. \( \exists y \in \mathbb{A} \), \( y > x \)

order of quantifiers is important!

A word about variables

\( x, y \) are assumed to be real unless otherwise specified in this course.
Negation of quantifiers is tricky!

In words, (assuming ~rainy = sunny)

Negation of “Every day is rainy” isn’t “Every day is sunny.”

It’s: “At least one day is sunny.”
Symbolically, negation of $\forall x, p(x)$ is $\exists x \ni \neg p(x)$

i.e. $\neg [\forall x, p(x)] \iff \exists x \ni \neg p(x)$

$\neg [\exists x \ni p(x)] \iff \forall x, \neg p(x)$

Ex Negate:

(a) $\forall x, g(x) > 0$  $\exists x \ni \neg [g(x) > 0]$
   $\exists x \ni g(x) \leq 0$

(b) $\exists x \ni f'(x) = 0$  $\forall x, f'(x) \neq 0$.

(c) $\forall x, (\exists y \ni y > x)$  $\exists x \ni \neg [\exists y \ni y > x]$
   $\exists x \ni \forall y, y \leq x$. 
Don’t go overboard!

Negate:

\[ \forall \varepsilon > 0 \exists \delta > 0 \text{ such that } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon \]

means: \[ \lim_{x \to a} f(x) = L \] (later this semester)

Negation is: \[ \lim_{x \to a} f(x) \neq L \]

There is a counterexample:

\[ \exists \varepsilon > 0 \text{ such that } \forall \delta > 0 \exists x \in [a - \delta, a + \delta] \text{ with } |x - a| < \delta \text{ and } |f(x) - f(a)| \geq \varepsilon \]