§2.2 Relations

Or: “Sophisticated definitions of things you (mostly) already know.”

5 five definitions in this section.

1. Ordered Pair
2. Cartesian Product
3. Relation
4. Equivalence Relation
5. (partition into) Equiv. Classes.
Sets are unordered - but often order matters!

With pts/vectors: \((1,2) \neq (2,1)\). \((3,4,5) \neq (5,4,3)\)

Option 1: Define a new “ordered set.”

Option 2: Mathematicians like building everything out of a few basic objects.
Def The ordered pair \((a,b)\) is the set
\[
(a, b) = \{\{a\}, \{a, b\}\} = \{\{a, b\}, \{a\}\} = \{\{b, a\}, \{a\}\}
\]

Ex \((1,2) = \{\{1\}, \{1, 2\}\}\) # 11
\((2,1) = \{\{2\}, \{2, 1\}\}\)

Thm \((a,b) = (c,d) \iff a=c \text{ and } b=d\)

Pf \(\iff\) Suppose \(a=c\) and \(b=d\). Then
\[
(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)
\]
\(\Rightarrow (\text{You try - book})\)
The \textbf{Cartesian Product} of sets $A, B$ is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

\textbf{Examples} points and vectors live in

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} = \{(x, y, z) \mid x, y \in \mathbb{R}^2, z \in \mathbb{R}\}$$

$$\text{or } \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$
\[ A = (2,5] \]
\[ B = (1,3] \]
\[ C = [5,7] \]
\[ D = [3,4] \]

\[ A \times B = (2,5] \times (1,3] = \{ (x,y) \mid 2 < x \leq 5 \text{ and } 1 < y \leq 3 \} \]

\[ A \times D = \{ (x,y) \mid x \in (2,5] \text{ and } y \in [3,4] \} \]

\[ C \times B = \{ (x,y) \mid x \in [5,7] \text{ and } y \in (1,3] \} \]

\[ C \times D = \{ (x,y) \mid 5 \leq x \leq 7 \text{ and } 3 \leq y \leq 4 \} \]
Relations
Often we're interested in relationships b/w elts of sets:

With #s: \(a < b, \quad x \geq y, \quad p = q\)
With shapes: \(\triangle ABC \sim \triangle DEF\) or \(\triangle ABC \sim \triangle DEF\) (or same area)

Technical Def

A relation between A and B is subset \(R \subset A \times B\). If \((a,b) \in R\) we write \(aRb\) (often replace \(R\) w/ symbol) and say “\(a\) is related to \(b\)”

Notes
1. If \(A = B\), we say \(R\) is relation on \(A\).
2. \(aRb \Rightarrow (a,b) \in R \subset A \times B \neq B \times A\)
   △ and \(b\) not nec. related to \(a\)!
Example: $P = \{A, B, C, D\}$

$S = \{\text{house, car, cat, dog, bike, grapes}\}$

A owns house \text{ ? } A, B married?

B owns house \text{ ? } A, B married?

C owns cat, dog, grapes

"owns" is a relation. As a subset of $P \times S$,

\[
\text{owns} = \{(A, \text{house}), (B, \text{house}), (C, \text{cat}), (C, \text{dog}), (C, \text{grapes})\}
\]

Notice: D owns nothing, nobody owns car, bike.
Ex = is a relation on \( \mathbb{Z} \), given by

\[ \{ ..., (-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), ... \} \subseteq \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 \]

Ex < is a rel'n on \( \mathbb{R} \), repr'd by

\[ < = \{ (0, 1), (0, 2), (0, 0.9), (0.09), ... \} \]

\[ = \{ (x, y) \in \mathbb{R}^2 \mid x < y \} \]

graphically
Ex. What relation on \( \mathbb{N} \) is represented in this picture?

\[ a \mid b \iff a \mid b, \text{ i.e. } a \text{ divides } b \text{ with no remainder} \]
Ex. Clock arithmetic, relation on \( \mathbb{Z} \).

Informally: every time we hit 12, we wrap back around to 0.

\[ a \equiv b \text{ or } a = b \mod 12 \text{ iff:} \]

1. \( a \) and \( b \) have same remainder when divided by 12:

\[
\begin{align*}
10 \div 12 &= 0 \ R \ 10 & 10 \equiv 22 \equiv 46 \\
22 \div 12 &= 1 \ R \ 10 \\
46 \div 12 &= 3 \ R \ 10
\end{align*}
\]

2. \( \exists k \in \mathbb{Z} \) such that \( a = b + 12k \)
Def an equivalence relation $R$ is on a set $S$ which satisfies these three conditions:

\[ \forall x, y, z \in S: \]
1. Reflexive: $x R x$
2. Symmetric: $x R y \Rightarrow y R x$
3. Transitive: $x R y \text{ and } y R z \Rightarrow x R z$

Ex: Which of these are equiv. relns?

- $R_1 = \text{YES!} \quad (\text{motivating example!})$
- $IN, < \quad \text{NO! not reflexive.} \quad (x \neq x)$
- $IN, \leq \quad \text{NO! reflexive, transitive, not symmetric.} \quad (y \leq 10, 10 \leq y)$
- $IN, = \quad \text{YES!}$
- $IN, \mid (\text{divides}) \quad \text{NO! reflexive, transitive, not symmetric.}$
- Polygons, \( \simeq \) (congruent) Yes
- Polygons, \( \sim \) (similar) Yes
- Lines, \( \parallel \) (parallel) Yes
- Lines, \( \perp \) (perp.) No! not reflexive: \( l \not\perp l \)

- \( \mathbb{Z}, \cong \) Yes! Let's prove the 3 conditions!

1. \textbf{Reflexive:} \( n \cong n \) \( \forall n \in \mathbb{Z} \).
   Let \( n \in \mathbb{Z} \). Then \( n = n + 12(0) \), so \( n \cong n \).

2. \textbf{Symmetric.}
   Suppose \( n \cong m \). Then \( n = m + 12k \), some \( k \in \mathbb{Z} \).
   Then \( n - 12k = m \), or \( m = n + 12(-k) \). Thus \( m \cong n \).

3. \textbf{Transitive.}
   Suppose \( n \cong m \), \( m \cong p \), so \( n = m + 12k \) and \( m = p + 12l \), \( k, l \in \mathbb{Z} \).
   Then \( n = (p + 12l) + 12k = p + 12(k + l) \), so \( n \cong p \).
Consider this relation on \(\mathbb{R}^2\):

\[(a, b) R (c, d) \text{ iff } a^2 + b^2 = c^2 + d^2\]

(a) Prove \(R\) is an equiv. reln. We need to show \(R\) is:

**Reflexive**

Let \((a, b) \in \mathbb{R}^2\). Then \(a^2 + b^2 = a^2 + b^2\), so \((a, b) R (a, b)\).

**Symmetric**

Let \((a, b)\) and \((c, d)\) be in \(\mathbb{R}^2\) such that \((a, b) R (c, d)\). Then \(a^2 + b^2 = c^2 + d^2\) and \(c^2 + d^2 = a^2 + b^2\) so \((c, d) R (a, b)\).

(because equality of \#s is symmetric)

**Transitive**

Let \((a, b) R (c, d)\) and \((c, d) R (e, f)\). Then \(a^2 + b^2 = c^2 + d^2\)

and \(c^2 + d^2 = e^2 + f^2\) (because equality of \#s is trans, \(a^2+b^2 = e^2+f^2\), and \((a, b) R (e, f)\).
More Abstract Example (like 2.2 #30)

Let \( S = \{a, b, c, d\} \). What is the eq. rel'n \( R \) on \( S \) with the fewest members such that \((a,b), (c,b) \in R\)?

\[ R \subseteq S \times S = \{(x,y) : x, y \in S \} \]

\[ R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (c,b), (b,a), (b,c), (a,c), (c,a)\} \]

\( R \) reflexive, given symm. trans

⚠️ We don't often do problems like this— but it's a good way to see how well you understand the technical definitions.

⚠️ In this example we use set-theoretic notation for rel'n, \( R \subseteq S \times S \), so elts of \( R \) were pairs: \( (a,b) \in R \iff aRb \)

In prev example, \((a,b)R(c,d) \iff (a,b), (c,d) \in R \subseteq R^2 \times R^2\)
Equivalence Reln's are our way of generalizing "equality" to other contexts, like geometry.

Given an equiv. reln, an important question is:

For a certain x, what is x related to??

Def: Given an eq. reln R on a set S, the equivalence class of \( x \in S \) is:

\[
E_x = \text{everything related (equiv) to } x
= \{ y : y \sim x \}
\]
$E_x = \{ y \in S : x \sim y \}$

**Ex Clock Arithmetic**

$E_0 = \{ \ldots, -12, 0, 12, 24, 36, 48, \ldots \}$

$E_1 = \{ \ldots, -11, 1, 13, 25, 37, 49, \ldots \}$  \quad =  \quad E_{-11} = E_{49} \quad \text{etc.}

$E_2 = \{ \ldots, -10, 2, 14, 26, 38, 50, \ldots \}$

$E_{-11} = \{ \ldots, -1, 11, 23, 35, \ldots \}$

$E_{12} = E_0$
$E_x = \{ y \in S : x \mathcal{R} y \}$

Ex: $2x + 3y = 6$ (← these x's, y's different than)

Equiv rel'n on lines in $\mathbb{R}^2$: $\parallel$

$E_x = \text{any line } \parallel l$

$= \text{any line w/ slope } -\frac{2}{3}$
Consider this relation on $\mathbb{R}^2$:

$$(a, b) R (c, d) \iff a^2 + b^2 = c^2 + d^2$$

(b) Describe the equivalence class of $(3,4)$. What does it look like geometrically?

$$E_{(3,4)} = \left\{ (x, y) : (x,y) \sim (3,4) \right\}$$

$$= \left\{ (x, y) : x^2 + y^2 = 3^2 + 4^2 \right\}$$

$$= \left\{ (x, y) : x^2 + y^2 = 5^2 \right\}$$

all pts on circle of rad 5, ctd @ origin.