§2.4 Cardinality

Ok, the Hotel Infinity teaches us that **infinity is weird**. Especially when we compare sizes of infinite sets.

For finite sets, it's easier! If \( A = \{1, 2, 3\} \) and \( B = \{0, \triangle, \square\} \) then \( A \) has 3 elts, \( B \) has 3, so \( B \) is "larger."
Ex. Write these sets in order from “smallest” (i.e. fewest members) to “largest” (most elements).

\[ N = \{1, 2, 3, \ldots \} \]
\[ N_0 = \{0, 1, 2, 3, \ldots \} \]
\[ Z = \{\ldots, -1, 0, 1, \ldots \} \]
\[ Q^+ = \{ \frac{p}{q} \geq 0, \text{ even} \} \]
\[ \mathbb{Q} \]
\[ \text{Irrationals} = \mathbb{R} \setminus \mathbb{Q} \]
\[ \mathbb{R} \]
\[ \mathbb{C} \]
\[ (0,1) \]
\[ [0,1] \]

all turn out to have

Same “size”

all same “size”

(larger than left column)
**Bijections** bring order to this chaos!

**Def.** Two sets $S, T$ are **equinumerous** if $\exists$ bijection $S \to T$. Write: $S \sim T$.

*idea* if $S \sim T$, they're in $1:1$ correspondence and are "same set" (with ells relabeled) hence same size

everything in $T$ is "hit" by exactly one elt in $S$
Ex. $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$

\[ f \text{ bijection } f : A \rightarrow B : \]
\[ f(1) = b \quad f \text{ is bijection "by inspection"}
\]
\[ f(2) = a \]
\[ f(3) = c \]
\[ \{1, 2, 3\} \sim \{a, b, c\} \]

Ex. $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$

No bijection possible so these sets have different “sizes”

Could choose different outputs for $f(1), f(2), f(3)$, but $f(4)$ will break injectivity.

(Google: Pigeonhole Principle)

Ex. $A = \{1, 2, 3, 4\}$ and $\mathbb{N}$

No bijection possible – we can choose up to 4 outputs but will never be surjective.
Ex \( \mathbb{N}_0 = \{ 0, 1, 2, 3, \ldots \} \) \( f: \mathbb{N}_0 \rightarrow \mathbb{N}, \ f(n) = n+1 \)

\( \mathbb{N} = \{ 1, 2, 3, \ldots \} \)

Is \( f \) surj? Let \( m \in \mathbb{N} \). Then
\[ m = f(m-1), \text{ and } m-1 \in \mathbb{N}_0. \]

Is \( f \) inj? Let \( n \neq m \) in \( \mathbb{N}_0 \). Then
\[ n+1 \neq m+1. \]

Ex \( \mathbb{N}, \mathbb{Z} \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) ) for ( f: \mathbb{N} \rightarrow \mathbb{Z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
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<tr>
<td>4</td>
<td>2</td>
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<tr>
<td>5</td>
<td>-2</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
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<td>\ldots</td>
<td>\ldots</td>
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\( bijn', \text{ so } \mathbb{N} \sim \mathbb{Z} \)
Ex \( \mathbb{N}, \mathbb{Q}^+ = \{ \frac{p}{q} : p, q \geq 20 \} \) (Cantor)

\[
\begin{align*}
&\text{start} \\
&\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \\
&\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \ldots \\
&\frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \ldots \\
&\vdots \\
\end{align*}
\]

We'll construct a bijection \( \mathbb{N} \rightarrow \mathbb{Q}^+ \) as follows:

Define \( f: \mathbb{N} \rightarrow \mathbb{Q}^+ \) as

\[
f(n) = n^{th} \text{ unique } \# \text{ we meet on this path.}
\]

\[
\begin{align*}
f(1) &= 0 \quad (= \frac{0}{1}) \\
f(2) &= 1 \quad (= \frac{1}{1}) \\
f(3) &= 2 \\
f(4) &= \frac{1}{2} \\
f(5) &= \frac{1}{3} \\
f(6) &= 3 \\
\end{align*}
\]

\( \mathbb{N} \sim \mathbb{Q}^+ \)
Def: A set $S$ is...

- **finite** if $S \sim \mathbb{I}_n = \{1, 2, 3, \ldots, n\}$
- **denumerable** if $S \sim \mathbb{N}$
- **countable** if $S$ is finite or denumerable

- **uncountable** if $S$ is not countable.

There’s a handy-dandy Venn Diagram in your book (p84)