Recall/Announcements.

HW posted later today

No quiz this week!

Wednesday: Work through whichever HW probs are trickiest - then fun stuff.

* We'll skip to Chapter 8 (Series) today, then retreat to Chapter 5 in December.
Cauchy Sequences

So far we've described convergence as els of a sequence (eventually) bunching up next to a limit.

\[
\text{Def: A seq } (s_n) \text{ of real #'s is a Cauchy Sequence if } \forall \varepsilon > 0 \exists N \text{ s.t. } n, m > N \Rightarrow |s_n - s_m| < \varepsilon
\]

i.e. eventually the #'s bunch up together
<table>
<thead>
<tr>
<th>2014 Fields Medalists</th>
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</thead>
<tbody>
<tr>
<td>Artur Avila</td>
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<tr>
<td>Manjul Bhargava</td>
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<tr>
<td>Martin Hairer</td>
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<td>Maryam Mirzakhani</td>
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</tbody>
</table>
Cauchy: $\forall \varepsilon > 0 \ \exists \ N \ s.t. \ n, m > N \Rightarrow |S_n - S_m| < \varepsilon$

$\text{Ex } S_n = \frac{(-1)^n}{n}$

Then $n, m > N$ (so both $\frac{1}{n}, \frac{1}{m} < \frac{3}{2}$) $\Rightarrow$

$|S_n - S_m| \leq |S_n| + |S_m| = \left| \frac{(-1)^n}{n} \right| + \left| \frac{(-1)^m}{m} \right| = \frac{1}{n} + \frac{1}{m} < \varepsilon$
Cauchy: \( \forall \varepsilon > 0 \ \exists N \text{ s.t. } n, m > N \Rightarrow |s_n - s_m| < \varepsilon \)

\[ t_n = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases} \]

This is not a Cauchy sequence. We can't force the #s to bunch up as needed in def.

Not Cauchy: \( \exists \varepsilon > 0 \text{ s.t. } \forall N, \exists n, m > N \text{ and } |s_n - s_m| \geq \varepsilon \).

Say \( \varepsilon = 1 \). For any \( N \), we can always find \( n, m > N \) with \( n \) odd, \( m \) even

\[ |s_n - s_m| = |(-1)^n - 1| = |-2| = 2 > \varepsilon. \]
Cauchy: \( \forall \varepsilon > 0 \ \exists N \text{ s.t. } n, m > N \implies |S_n - S_m| < \varepsilon \)

Why do we care?

Thm (\( S_n \)) converges \( \iff \) (\( S_n \)) Cauchy

Pf \( \leq \) not in this course.

\( \implies \) Suppose \( S_n \to s \), let \( \varepsilon > 0 \) be given.

Must show \( \exists N \text{ s.t. } n, m > N \implies |S_n - S_m| < \varepsilon \).

We know \( \exists N \text{ s.t. } n > N \implies |S_n - s| < \varepsilon/2 \).

Then \( n, m > N \) gives

\[
|S_n - S_m| = |S_n - S - S_m + S| = |S_n - S - (S_m - S)|
\]

\[
\leq |S_n - s| + |S_m - s| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]